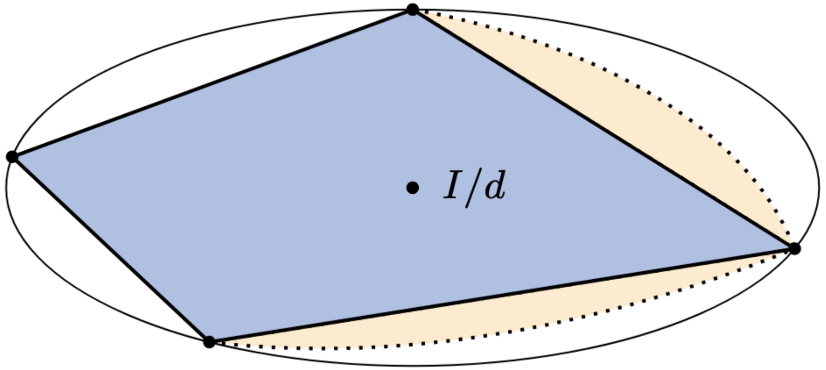


Characterization of Bound Magic States via the Kirkwood-Dirac Distribution

Peter (Songqinghao) Yang

Joint work with Jonathan J. Thio, Nicole Yunger Halpern, Stephan De Bievre, Crispin H. W. Barnes, and David R. M. Arvidsson-Shukur

Cavendish Laboratory, University of Cambridge

What are magic states?

What are magic states?

Stabilizer states

Clifford Operation

What are magic states?

e.g.

$$Z|0\rangle = |0\rangle$$

$$\mathcal{S} = \langle Z \rangle$$

Stabilizer states

Clifford Operation

What are magic states?

e.g.

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$Z|0\rangle = |0\rangle$$

$$\mathcal{S} = \langle Z_1 Z_2,\ X_1 X_2\rangle$$

$$\mathcal{S} = \langle Z\rangle$$

$$Z_1 Z_2 |\Phi^+\rangle = |\Phi^+\rangle \quad X_1 X_2 |\Phi^+\rangle = |\Phi^+\rangle$$

Stabilizer states

Clifford Operation

What are magic states?

e.g.

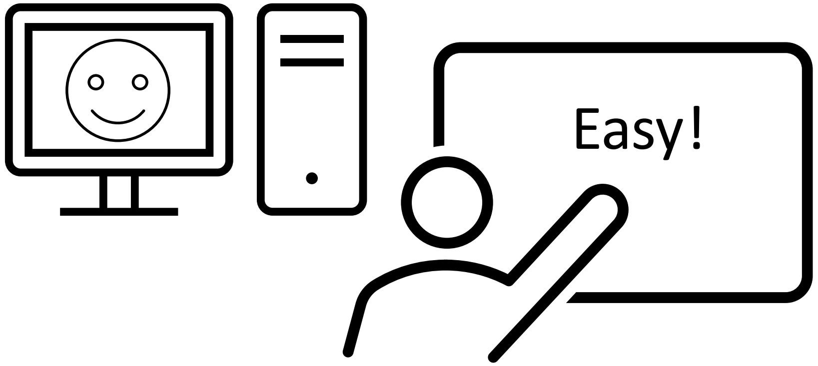
$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$Z|0\rangle = |0\rangle \qquad \mathcal{S} = \langle Z_1 Z_2,\, X_1 X_2 \rangle$$

$$\mathcal{S} = \langle Z \rangle \qquad Z_1 Z_2 |\Phi^+\rangle = |\Phi^+\rangle \quad X_1 X_2 |\Phi^+\rangle = |\Phi^+\rangle$$

Stabilizer states

Clifford Operation



Gottesman & Knill

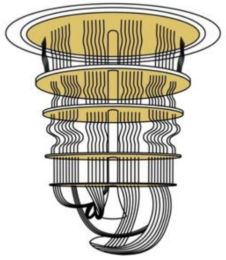
What are magic states?

e.g.

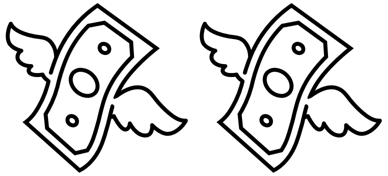
$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

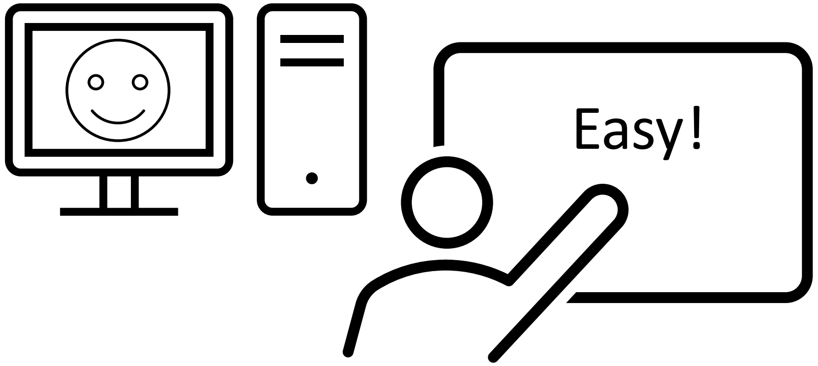
$$Z|0\rangle = |0\rangle \qquad \mathcal{S} = \langle Z_1 Z_2,\, X_1 X_2 \rangle$$

$$\mathcal{S} = \langle Z \rangle \qquad Z_1 Z_2 |\Phi^+\rangle = |\Phi^+\rangle \quad X_1 X_2 |\Phi^+\rangle = |\Phi^+\rangle$$



Stabilizer states | Clifford Operation **+** [ ] **=** Universal QC





Gottesman & Knill

What are magic states?

Pure non-stabilizer states

e.g.

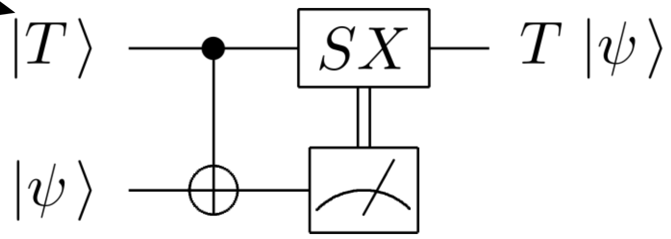
$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$Z|0\rangle = |0\rangle \qquad \mathcal{S} = \langle Z_1 Z_2,\ X_1 X_2\rangle$$

$$\mathcal{S} = \langle Z\rangle \qquad Z_1 Z_2 |\Phi^+\rangle = |\Phi^+\rangle \quad X_1 X_2 |\Phi^+\rangle = |\Phi^+\rangle$$

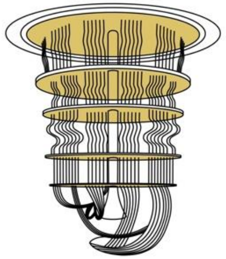
Stabilizer states | **Clifford Operation**

+

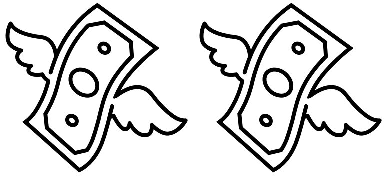


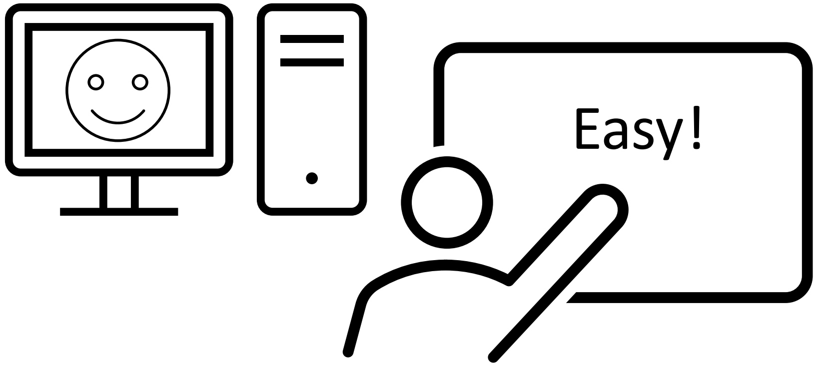
Magic State Injection

=



Universal QC





Gottesman & Knill

What are magic states?

Pure non-stabilizer states

e.g.

$$Z|0\rangle = |0\rangle$$

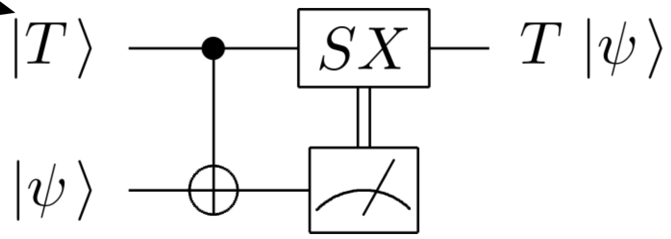
$$\mathcal{S} = \langle Z \rangle$$

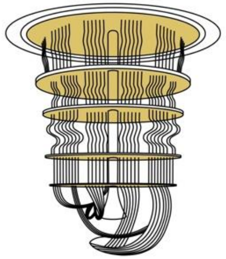
$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

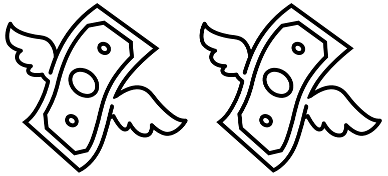
$$\mathcal{S} = \langle Z_1 Z_2,\ X_1 X_2 \rangle$$

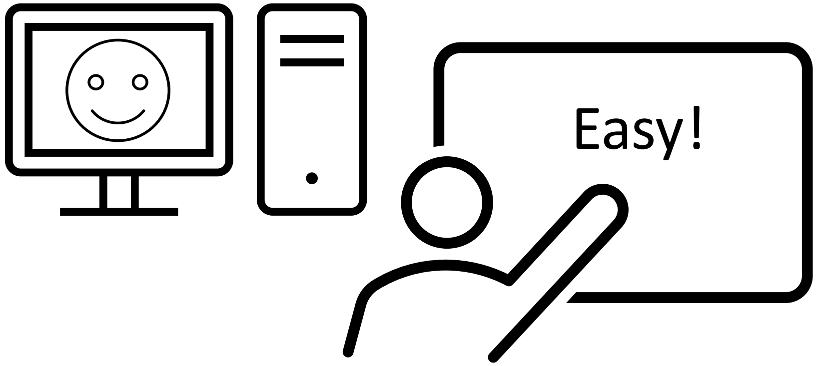
$$Z_1 Z_2 |\Phi^+\rangle = |\Phi^+\rangle \quad X_1 X_2 |\Phi^+\rangle = |\Phi^+\rangle$$

Stabilizer states + Clifford Operation + Magic State Injection = Universal QC

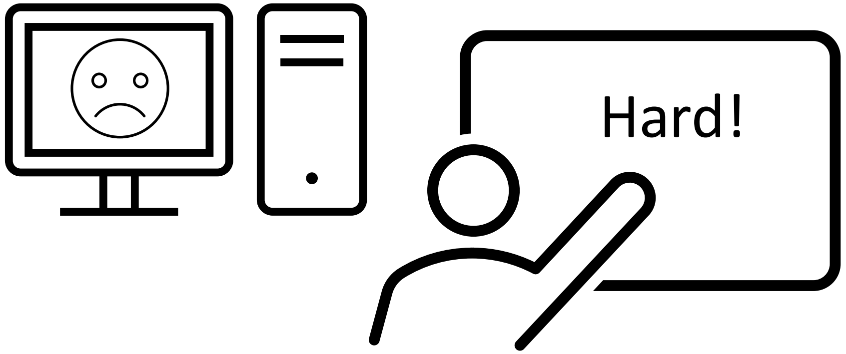




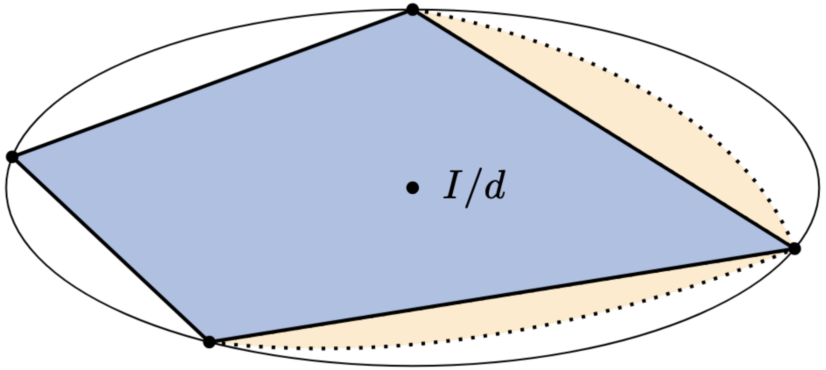




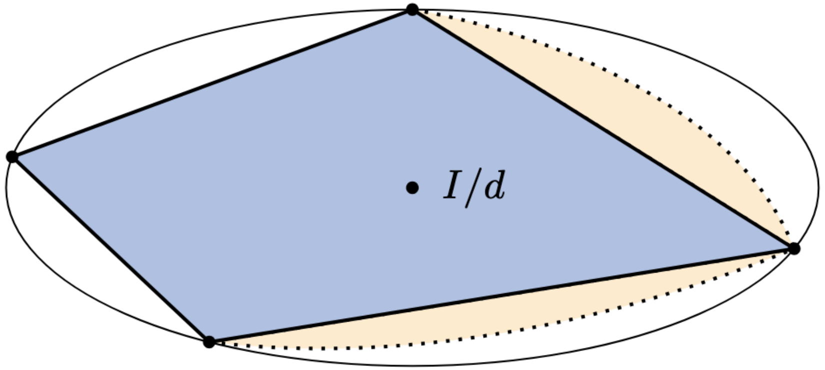
Gottesman & Knill



What are bound magic states?

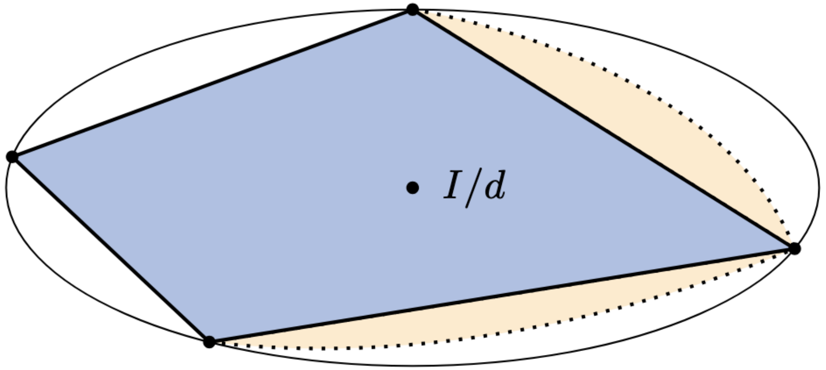


What are bound magic states?



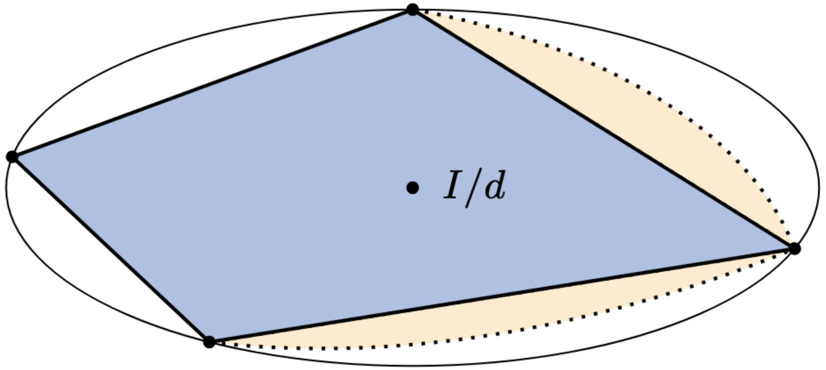
- Bravyi & Kitaev: Most non-stabilizer states are distillable magic states

What are bound magic states?



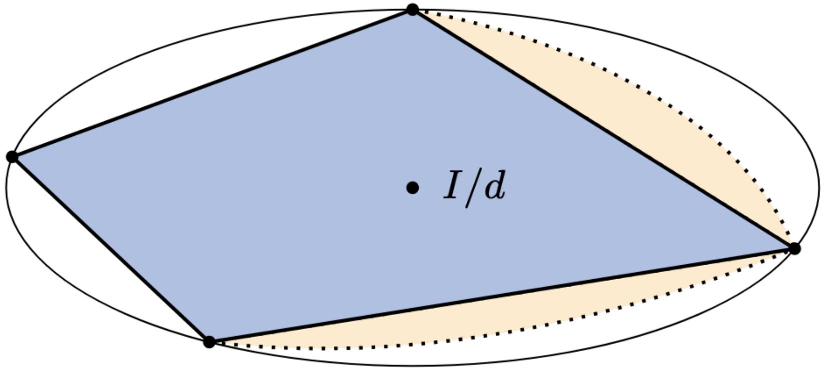
- Bravyi & Kitaev: Most non-stabilizer states are distillable magic states
- Campbell & Browne: There exist non-distillable, non-stabilizer states

What are bound magic states?



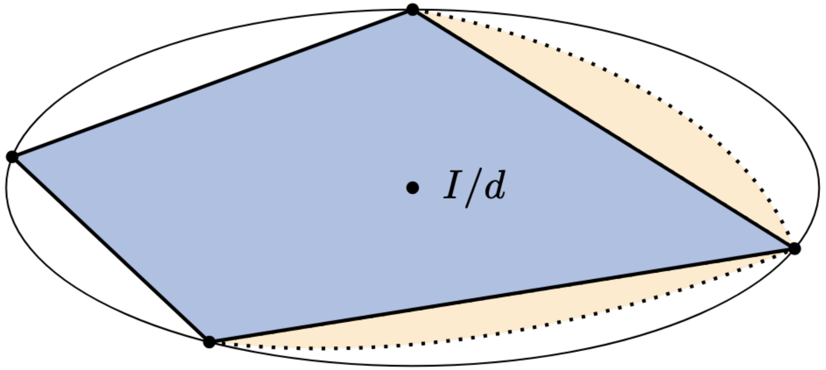
- Bravyi & Kitaev: Most non-stabilizer states are distillable magic states
- Campbell & Browne: There exist non-distillable, non-stabilizer states
- Veitch, Gross: There exist bound magic state in odd dimensions

What are bound magic states?



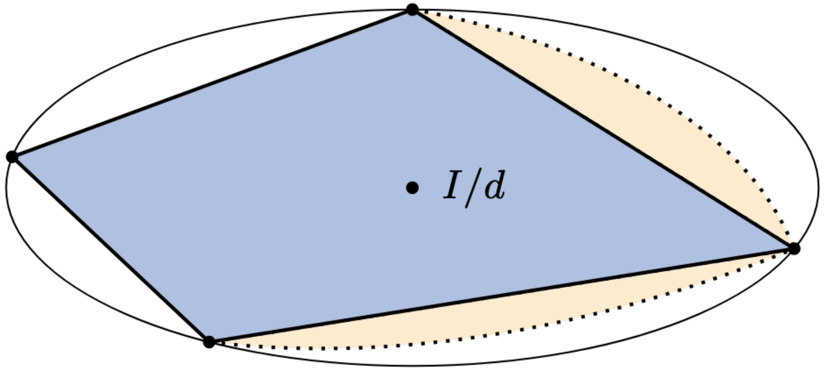
- Bravyi & Kitaev: Most non-stabilizer states are distillable magic states
- Campbell & Browne: There exist non-distillable, non-stabilizer states
- Veitch, Gross: There exist bound magic state in odd dimensions
- Zurel: Pushed the idea to even dimensions

What are bound magic states?



- Bravyi & Kitaev: Most non-stabilizer states are distillable magic states
- Campbell & Browne: There exist non-distillable, non-stabilizer states
- Veitch, Gross: There exist bound magic state in odd dimensions
- Zurel: Pushed the idea to even dimensions
- Why bound magic states? Knowing more bound magic states, we extend known classical simulability

What are bound magic states?

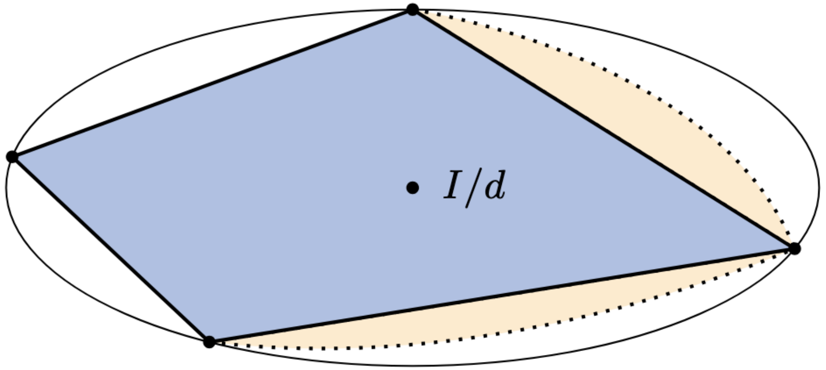


- Bravyi & Kitaev: Most non-stabilizer states are distillable magic states
- Campbell & Browne: There exist non-distillable, non-stabilizer states
- Veitch, Gross: There exist bound magic state in odd dimensions
- Zurel: Pushed the idea to even dimensions
- Why bound magic states? Knowing more bound magic states, we extend known classical simulability

Supercomputer   Operation : Cliffords

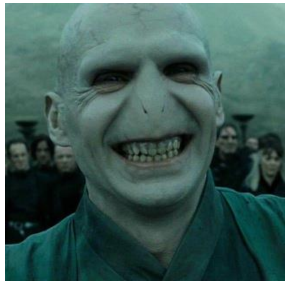
Quantum Computational Power

What are bound magic states?



- Bravyi & Kitaev: Most non-stabilizer states are distillable magic states
- Campbell & Browne: There exist non-distillable, non-stabilizer states
- Veitch, Gross: There exist bound magic state in odd dimensions
- Zurel: Pushed the idea to even dimensions
- Why bound magic states? Knowing more bound magic states, we extend known classical simulability

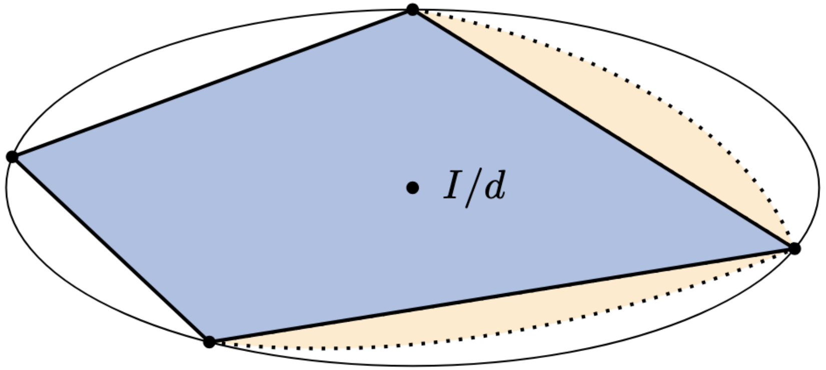
Supercomputer



Operation : Cliffords

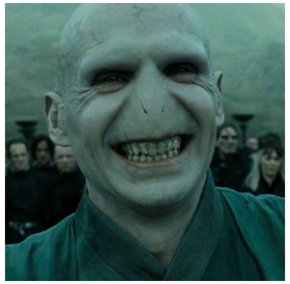
Quantum Computational Power

What are bound magic states?



- Bravyi & Kitaev: Most non-stabilizer states are distillable magic states
- Campbell & Browne: There exist non-distillable, non-stabilizer states
- Veitch, Gross: There exist bound magic state in odd dimensions
- Zurel: Pushed the idea to even dimensions
- Why bound magic states? Knowing more bound magic states, we extend known classical simulability

Supercomputer



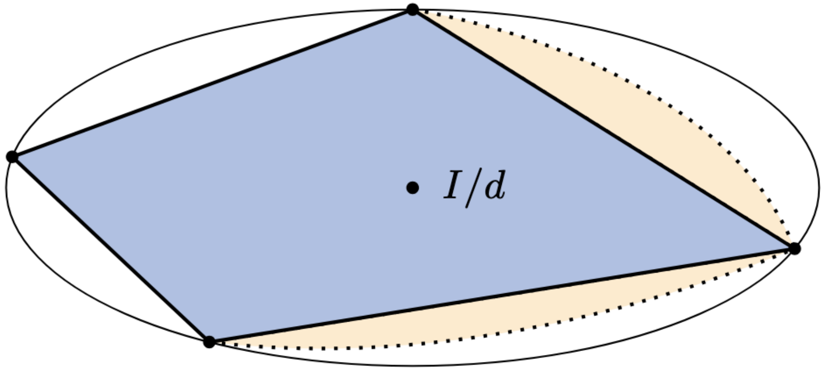
Operation : Cliffords

Quantum Computational Power

Stabilizer States

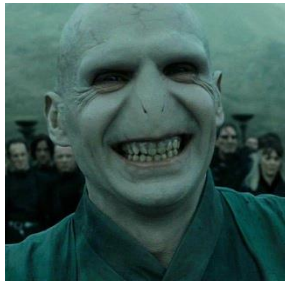
Muggles

What are bound magic states?

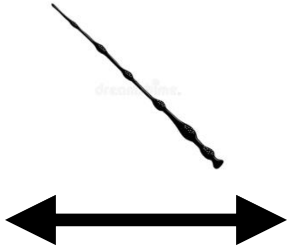


- Bravyi & Kitaev: Most non-stabilizer states are distillable magic states
- Campbell & Browne: There exist non-distillable, non-stabilizer states
- Veitch, Gross: There exist bound magic state in odd dimensions
- Zurel: Pushed the idea to even dimensions
- Why bound magic states? Knowing more bound magic states, we extend known classical simulability

Supercomputer



Operation : Cliffords



Quantum Computational Power

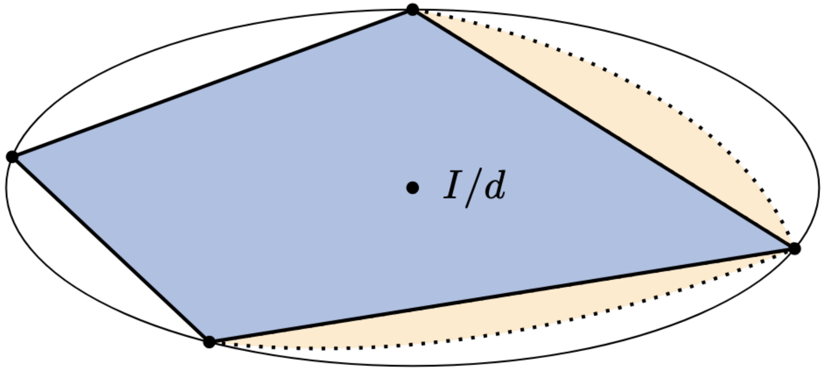
Magic State

Wizards&Witches

Stabilizer States

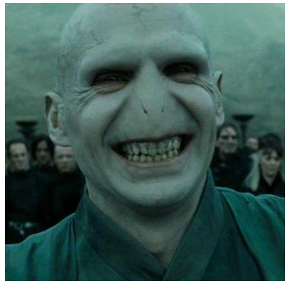
Muggles

What are bound magic states?



- Bravyi & Kitaev: Most non-stabilizer states are distillable magic states
- Campbell & Browne: There exist non-distillable, non-stabilizer states
- Veitch, Gross: There exist bound magic state in odd dimensions
- Zurel: Pushed the idea to even dimensions
- Why bound magic states? Knowing more bound magic states, we extend known classical simulability

Supercomputer



Operation : Cliffords

Quantum Computational Power

Magic State

Wizards&Witches

Bound Magic State

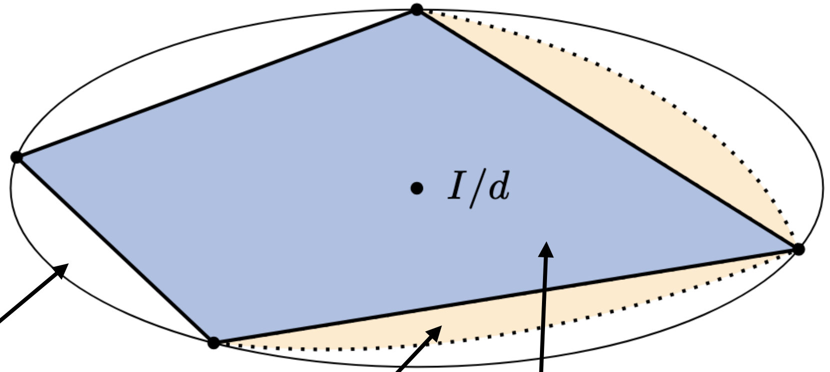
Squibs

Stabilizer States

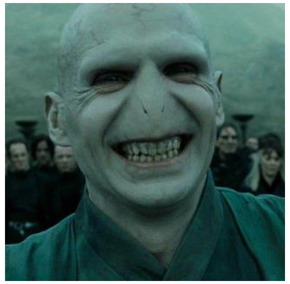
Muggles

What are bound magic states?

- Bravyi & Kitaev: Most non-stabilizer states are distillable magic states
- Campbell & Browne: There exist non-distillable, non-stabilizer states
- Veitch, Gross: There exist bound magic state in odd dimensions
- Zurel: Pushed the idea to even dimensions
- Why bound magic states? Knowing more bound magic states, we extend known classical simulability



Supercomputer



Operation : Cliffords

Quantum Computational Power

Magic State

Wizards&Witches

Bound Magic State

Squibs

Stabilizer States

Muggles

How to find bound magic states?

How to find bound magic states?

Stabilizer States

+

Clifford Operation

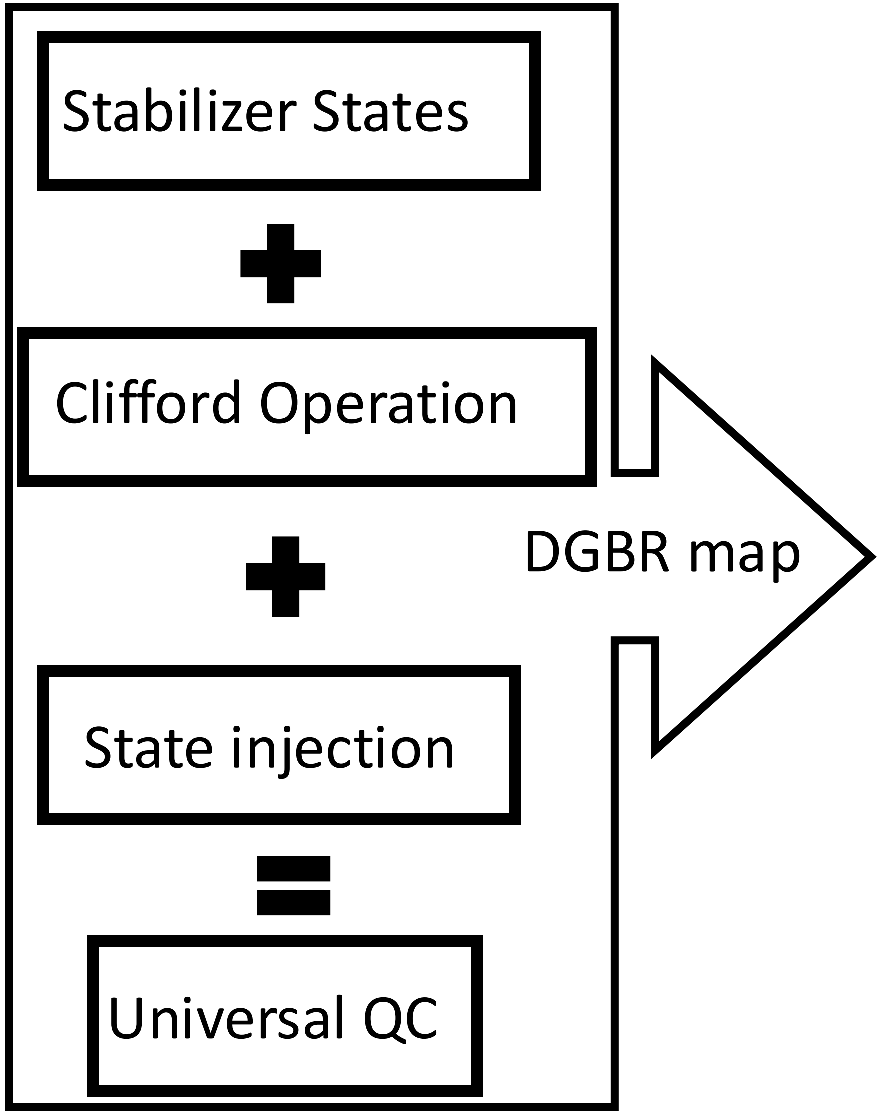
+

State injection

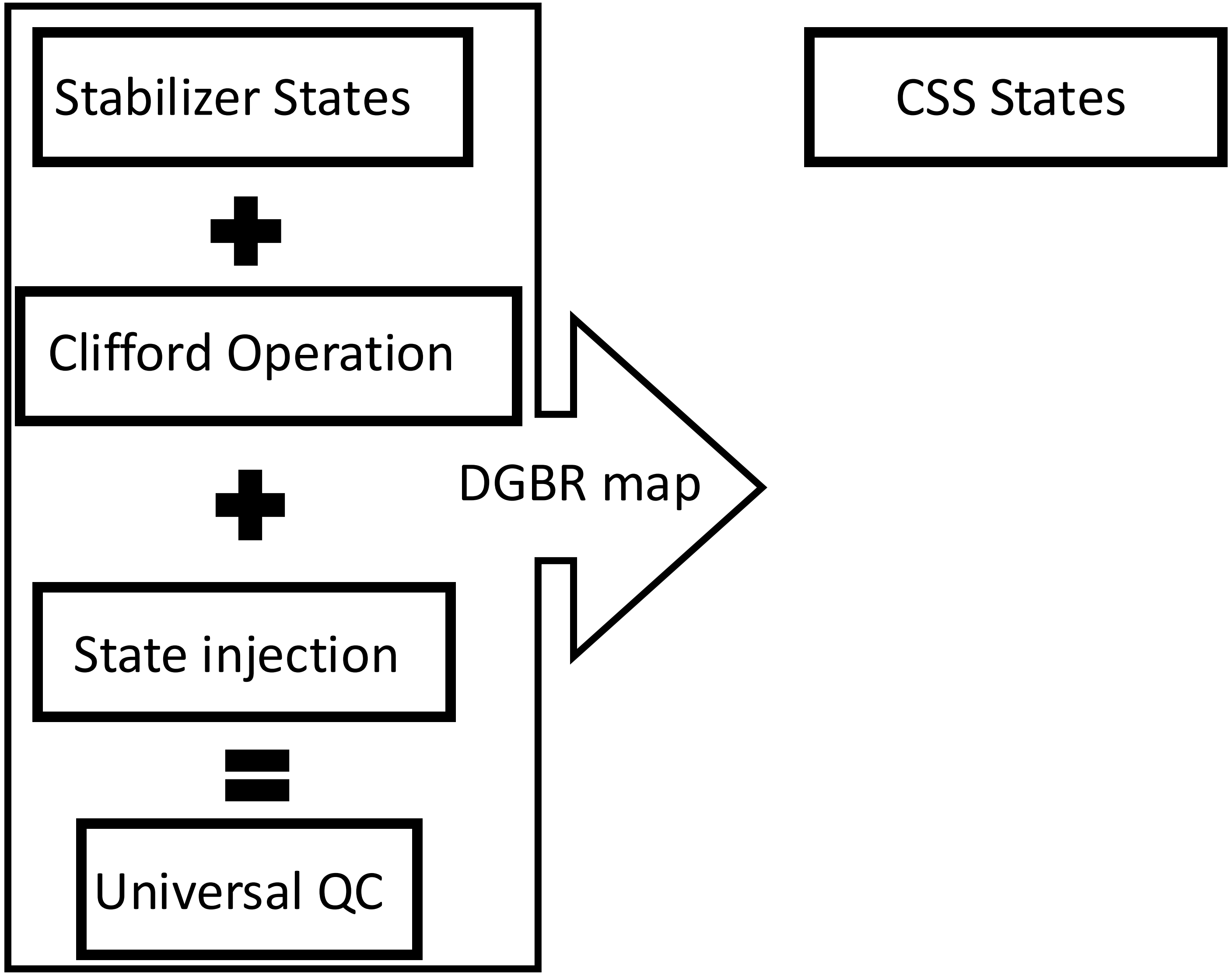
=

Universal QC

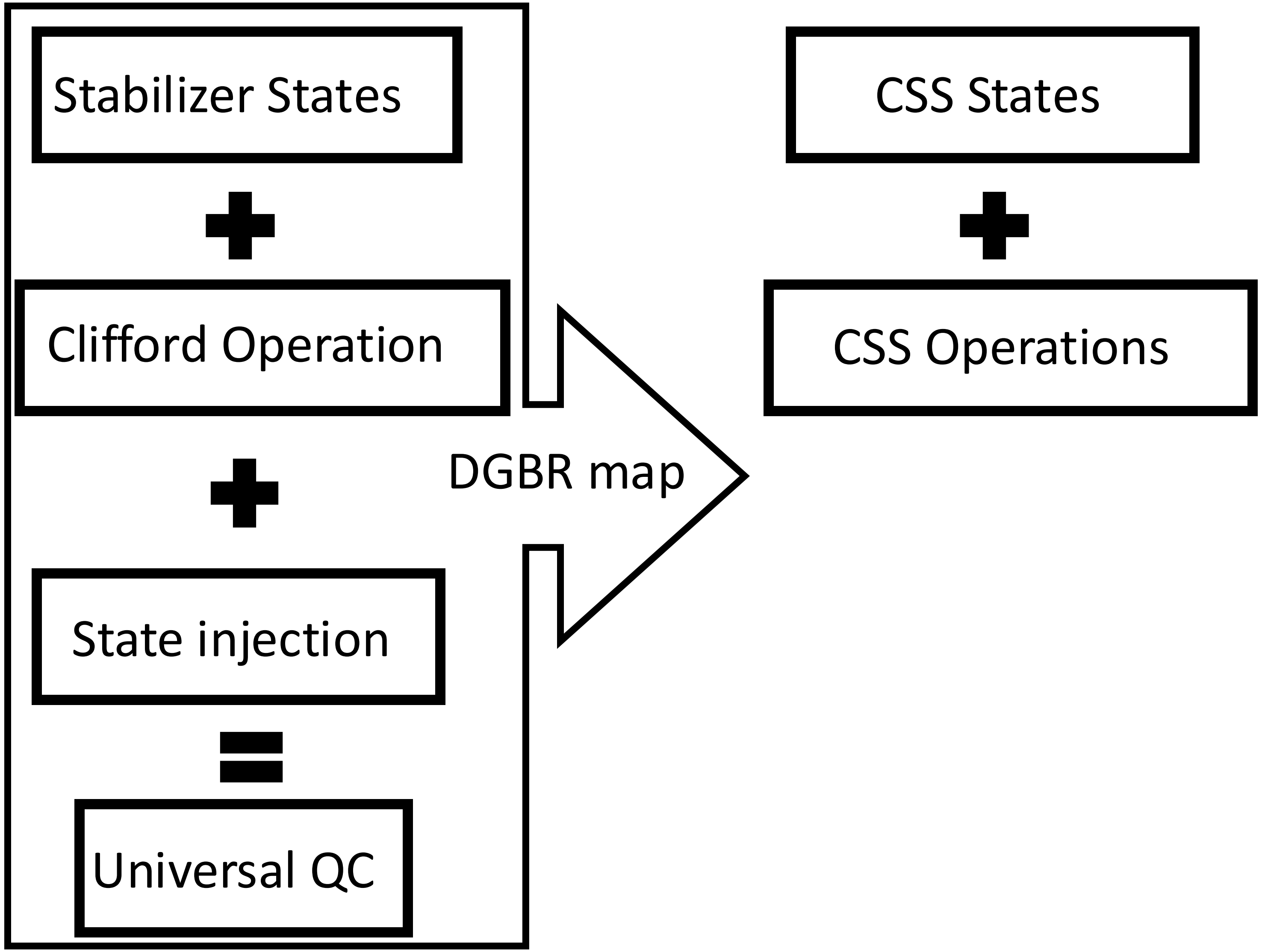
How to find bound magic states?



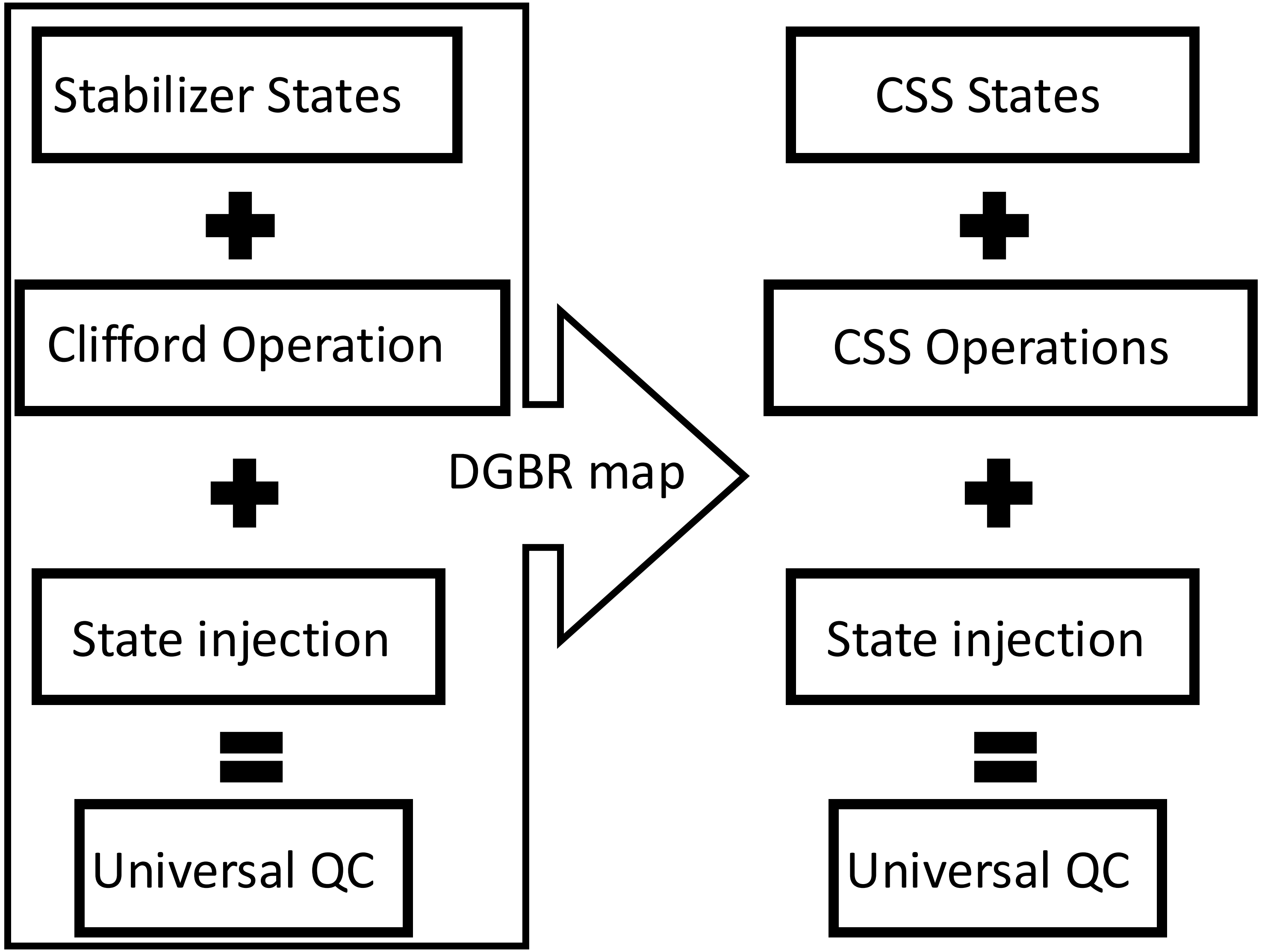
How to find bound magic states?



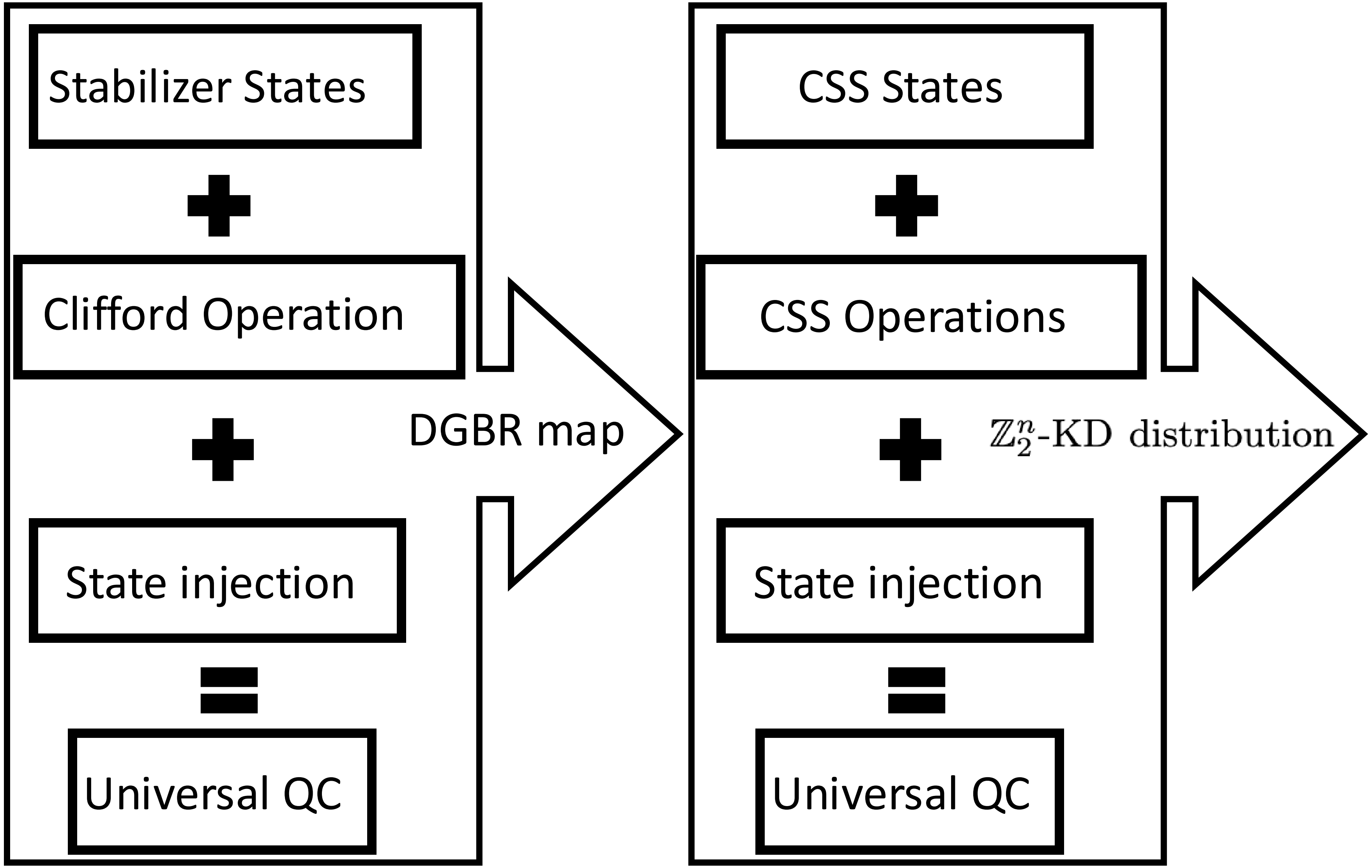
How to find bound magic states?



How to find bound magic states?



How to find bound magic states?



How to find bound magic states?

Stabilizer States

➕

Clifford Operation

➕

State injection

=

Universal QC

DGBR map

CSS States

➕

CSS Operations

➕

State injection

=

Universal QC

$\mathbb{Z}_2^n$-KD distribution

KD positive & Stabilizer

Muggles

How to find bound magic states?

Stabilizer States

+

Clifford Operation

+

State injection

=

Universal QC

DGBR map

CSS States

+

CSS Operations

+

State injection

=

Universal QC

$\mathbb{Z}_2^n$-KD distribution

KD positive & Stabilizer

Muggles

KD negative
Non-Stabilizer

Wizards&Witches

How to find bound magic states?

Stabilizer States

+

Clifford Operation

+

State injection

=

Universal QC

DGBR map

CSS States

+

CSS Operations

+

State injection

=

Universal QC

$\mathbb{Z}_2^n$-KD distribution

KD positive & Stabilizer

Muggles

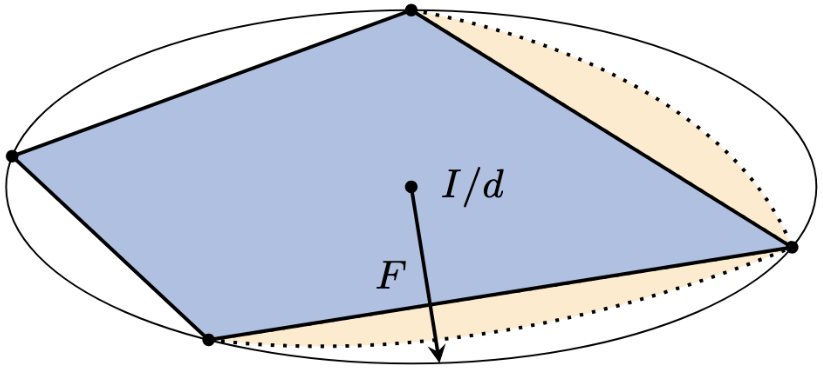
KD negative
Non-Stabilizer

Wizards&Witches

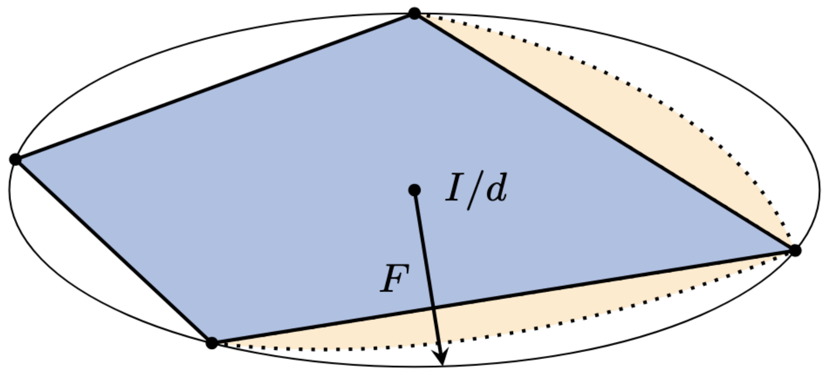
KD positive
Non-Stabilizer

Squibs

Where are bound magic states?



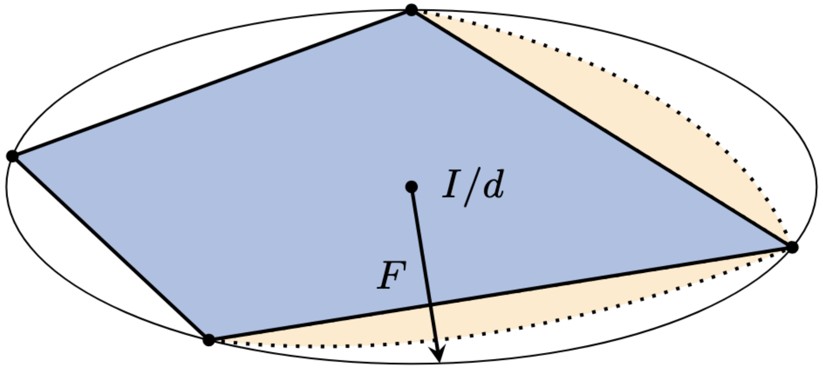
Where are bound magic states?



Example:

Let's choose a direction orthogonal to a facet of
the CSS polytope and to a real ridge of the stabilizer polytope

Where are bound magic states?

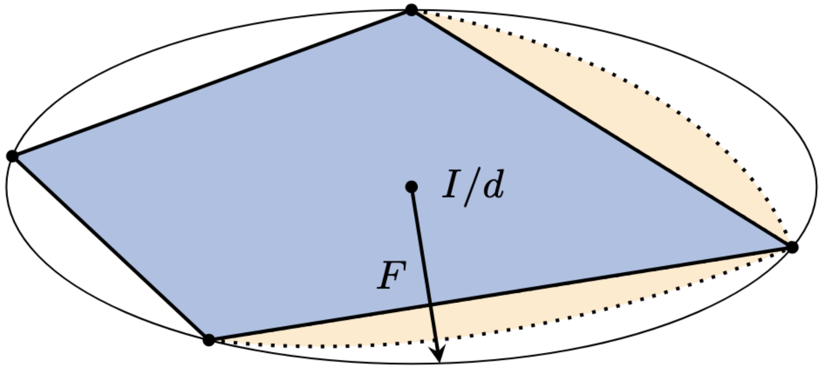


Example:

Let's choose a direction orthogonal to a facet of
the CSS polytope and to a real ridge of the stabilizer polytope

$$F = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & -1 & -1 & -2 \\ 1 & -1 & -2 & -1 \end{pmatrix}$$

Where are bound magic states?



Example:

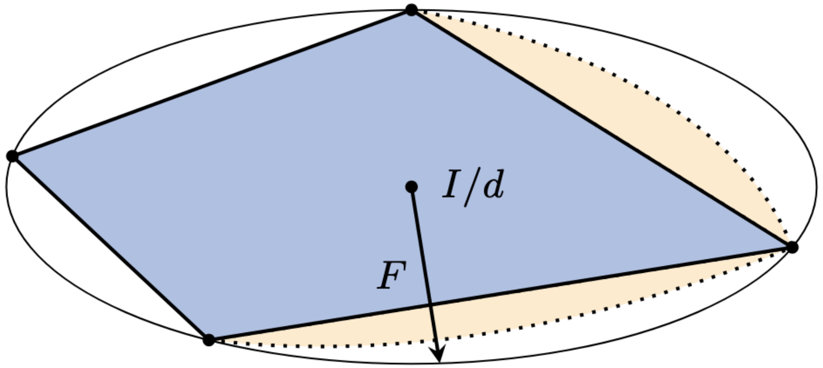
Let's choose a direction orthogonal to a facet of
the CSS polytope and to a real ridge of the stabilizer polytope

$$F = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & -1 & -1 & -2 \\ 1 & -1 & -2 & -1 \end{pmatrix}$$

Construct a state

$$\rho_\lambda = \frac{1}{4} I + \lambda F$$

Where are bound magic states?



Example:

Let's choose a direction orthogonal to a facet of
the CSS polytope and to a real ridge of the stabilizer polytope

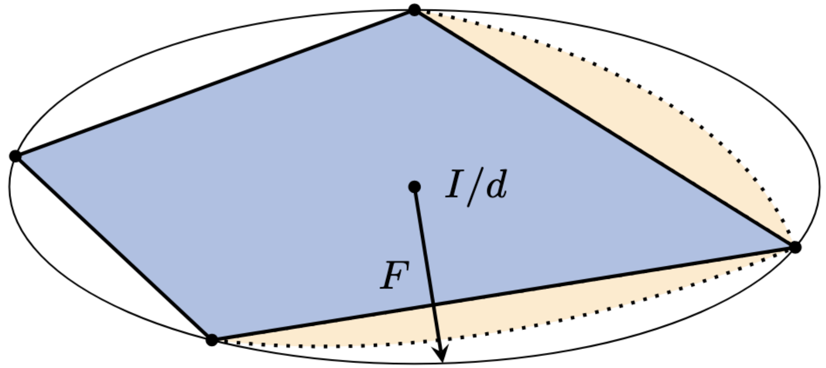
$$F = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & -1 & -1 & -2 \\ 1 & -1 & -2 & -1 \end{pmatrix}$$

Construct a state

$$\rho_\lambda = \frac{1}{4} I + \lambda F$$

ρ_λ is KD-positive if $\lambda \in [0, 1/(4+8\sqrt{2})]$

Where are bound magic states?



Example:

Let's choose a direction orthogonal to a facet of
the CSS polytope and to a real ridge of the stabilizer polytope

$$F = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & -1 & -1 & -2 \\ 1 & -1 & -2 & -1 \end{pmatrix}$$

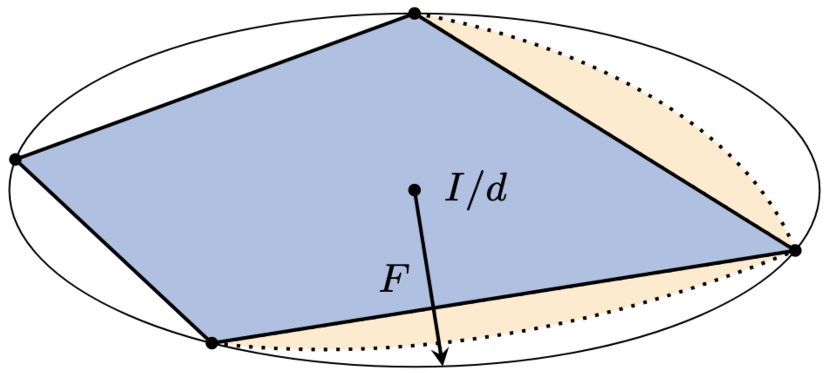
Construct a state

$$\rho_\lambda = \frac{1}{4} I + \lambda F$$

ρ_λ is KD-positive if $\lambda \in [0, 1/(4+8\sqrt{2})]$

lies outside the rebit stabilizer polytope for $\lambda > 1/20$

Where are bound magic states?



Example:

Let's choose a direction orthogonal to a facet of
the CSS polytope and to a real ridge of the stabilizer polytope

$$F = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & -1 & -1 & -2 \\ 1 & -1 & -2 & -1 \end{pmatrix}$$

Construct a state

$$\rho_\lambda = \frac{1}{4} I + \lambda F$$

ρ_λ is KD-positive if $\lambda \in [0, 1/(4+8\sqrt{2})]$

lies outside the rebit stabilizer polytope for $\lambda > 1/20$

Bound magic state if $\lambda \in (1/20, 1/(4+8\sqrt{2})]$

Where are bound magic states?

Example:

Let's choose a direction orthogonal to a facet of
the CSS polytope and to a real ridge of the stabilizer polytope

$$F = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & -1 & -1 & -2 \\ 1 & -1 & -2 & -1 \end{pmatrix}$$

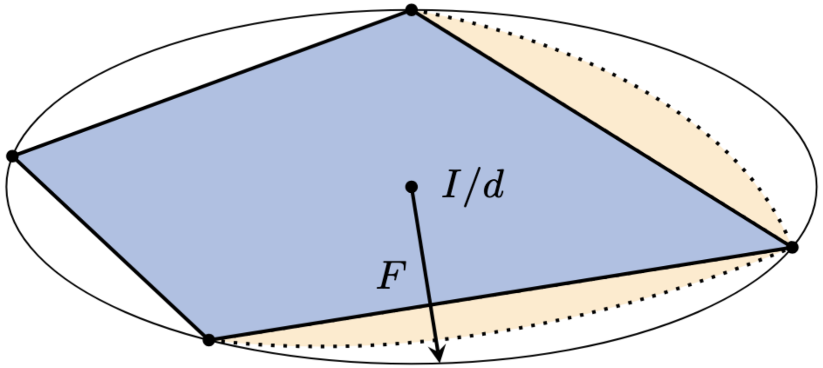
Construct a state

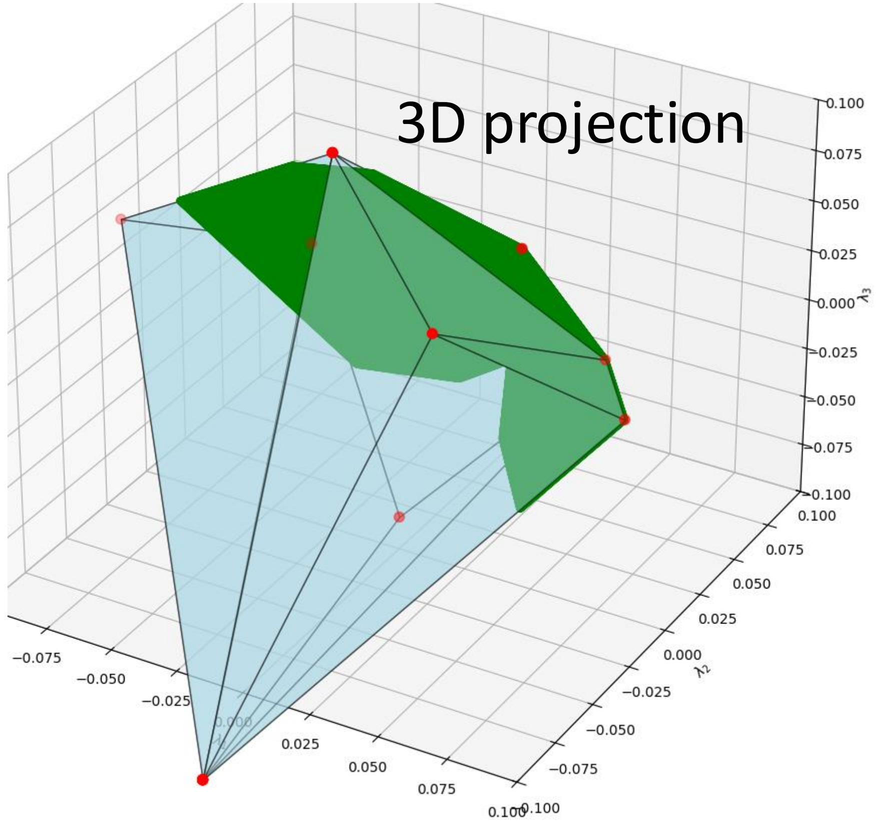
$$\rho_\lambda = \frac{1}{4} I + \lambda F$$

ρ_λ is KD-positive if $\lambda \in [0, 1/(4+8\sqrt{2})]$

lies outside the rebit stabilizer polytope for $\lambda > 1/20$

Bound magic state if $\lambda \in (1/20, 1/(4+8\sqrt{2})]$





Where are bound magic states?

Example:

Let's choose a direction orthogonal to a facet of
the CSS polytope and to a real ridge of the stabilizer polytope

$$F = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & -1 & -1 & -2 \\ 1 & -1 & -2 & -1 \end{pmatrix}$$

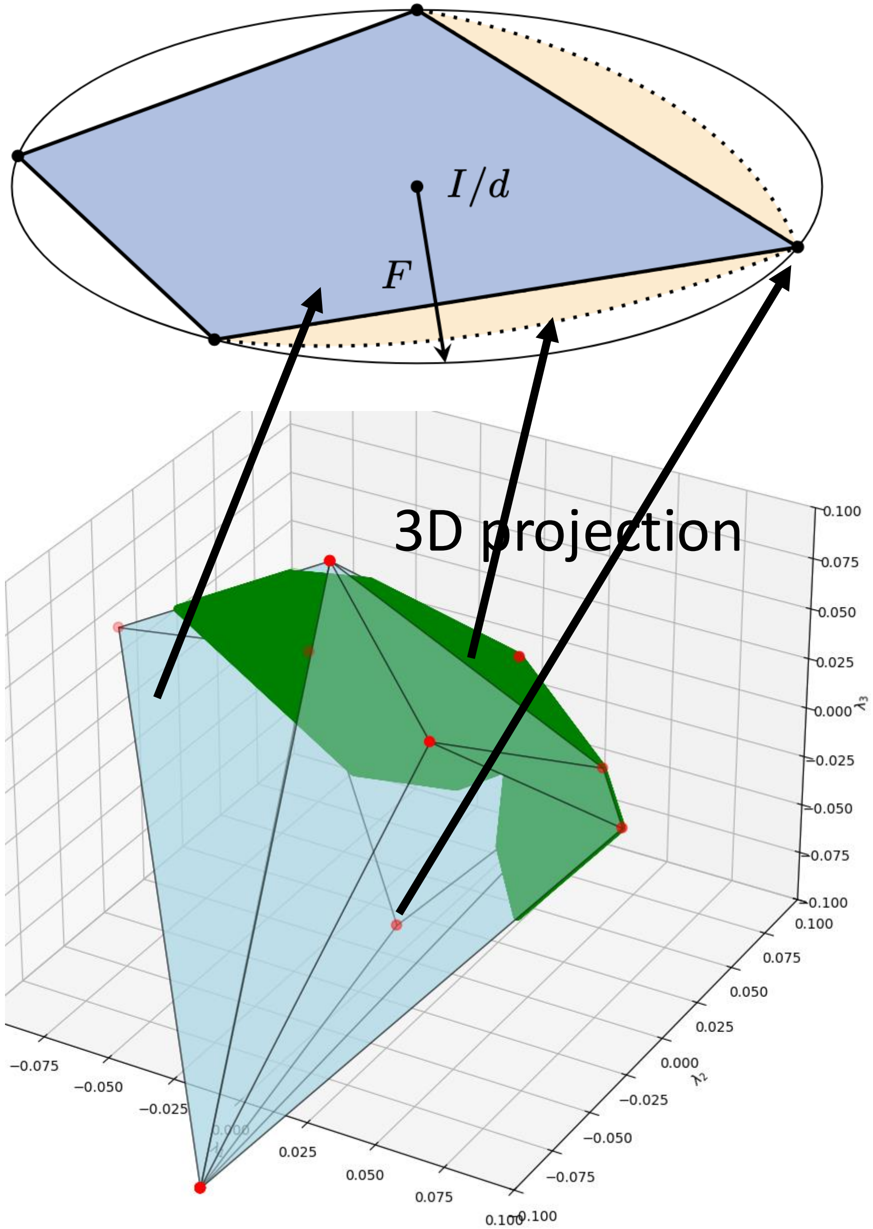
Construct a state

$$\rho_\lambda = \frac{1}{4}I + \lambda F$$

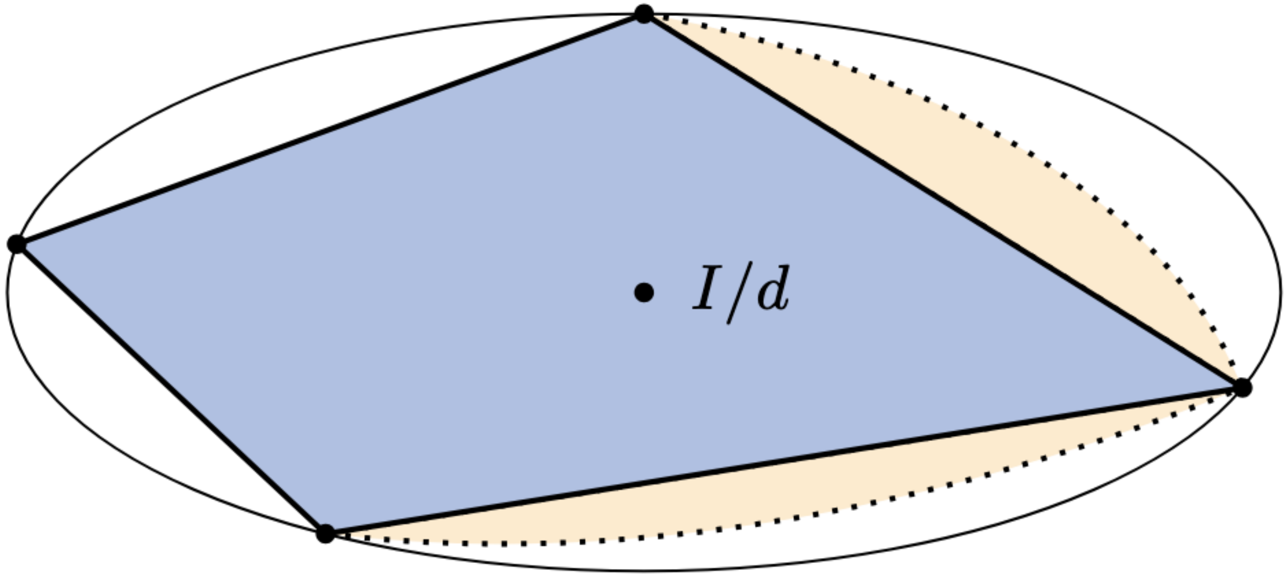
ρ_λ is KD-positive if $\lambda \in [0, 1/(4+8\sqrt{2})]$

lies outside the rebit stabilizer polytope for $\lambda > 1/20$

Bound magic state if $\lambda \in (1/20, 1/(4+8\sqrt{2})]$



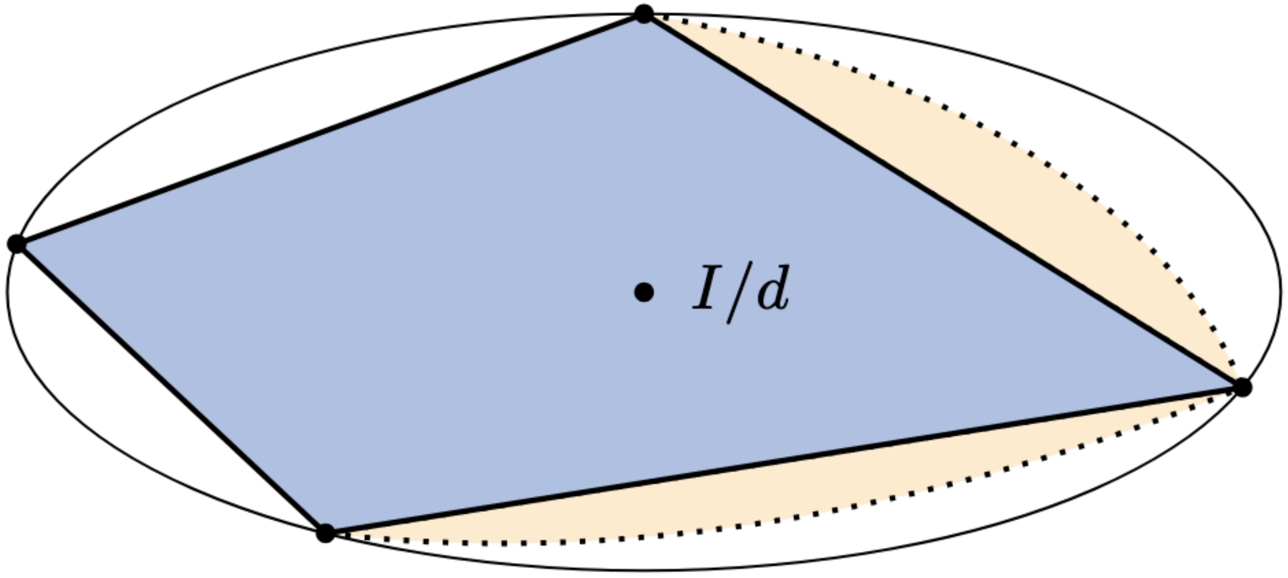
How many bound magic states are there?



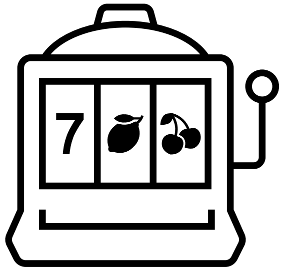
How many bound magic states are there?

Sampling a Random Rebit
Density Matrix via Ginibre Ensemble

$$\rho = \frac{AA^\top}{\mathrm{Tr}[AA^\top]}$$

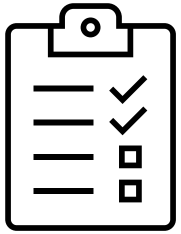


How many bound magic states are there?



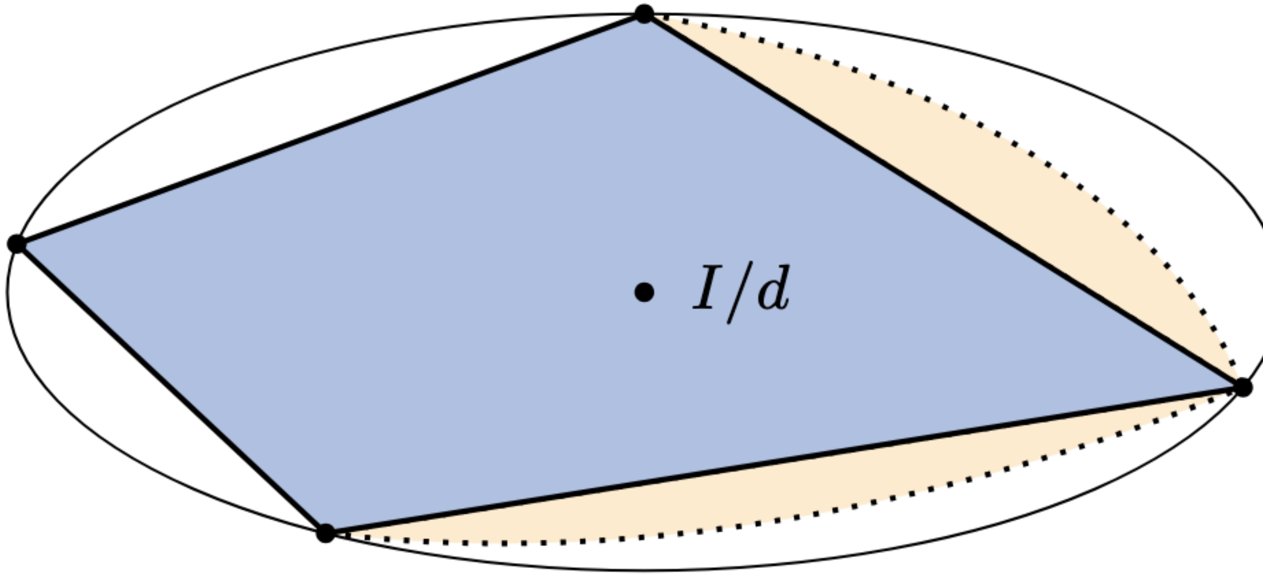
Sampling a Random Rebit
Density Matrix via Ginibre Ensemble

$$\rho = \frac{AA^\top}{\mathrm{Tr}[AA^\top]}$$

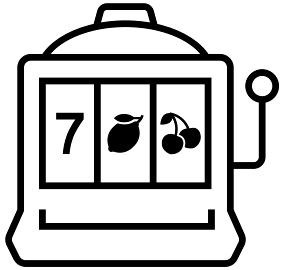


Are there any negative entries in the KD distribution?

Is this state a stabilizer state? (linear programming)

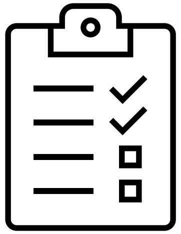


How many bound magic states are there?



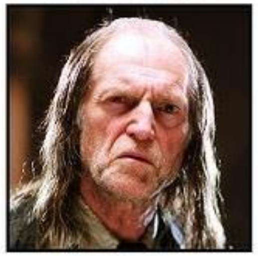
Sampling a Random Rebit
Density Matrix via Ginibre Ensemble

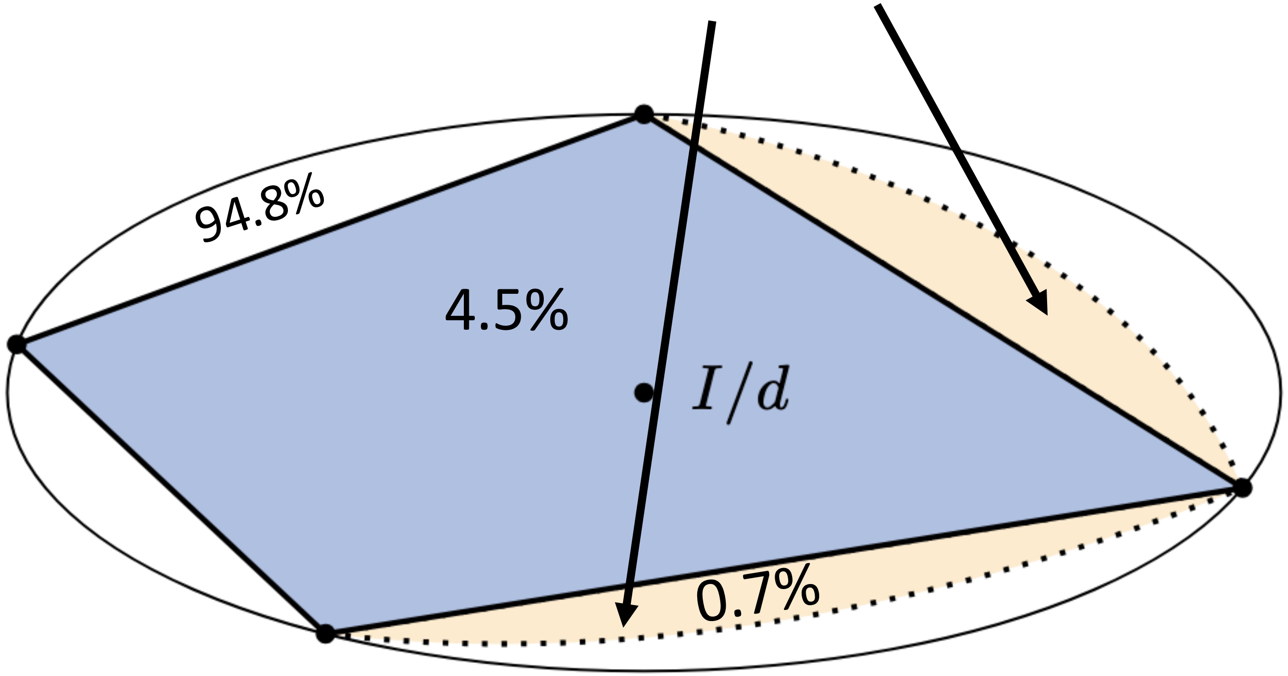
$$\rho = \frac{AA^\top}{\mathrm{Tr}[AA^\top]}$$



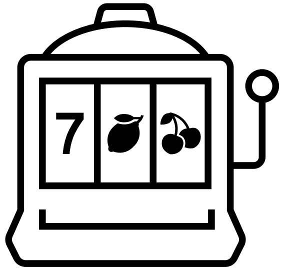
Are there any negative entries in the KD distribution?

Is this state a stabilizer state? (linear programming)



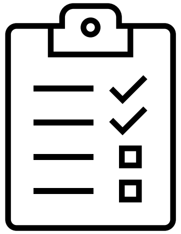


How many bound magic states are there?



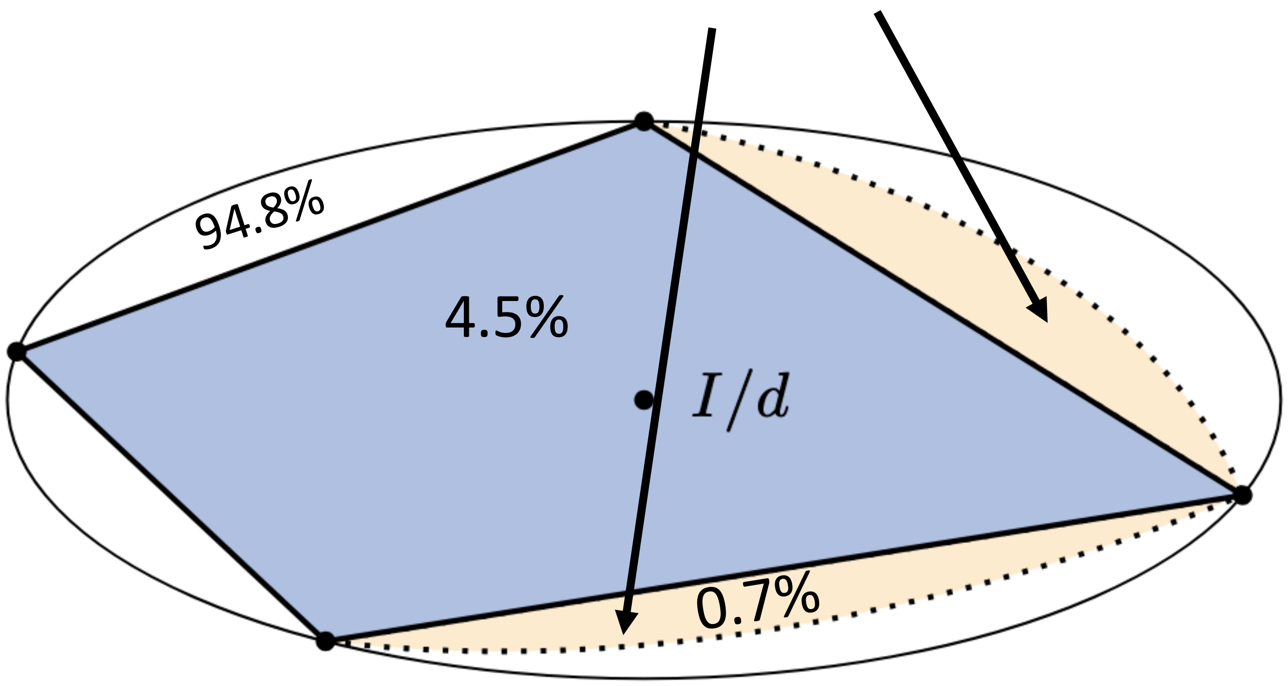
Sampling a Random Rebit Density Matrix via Ginibre Ensemble

$$\rho = \frac{AA^\top}{\mathrm{Tr}[AA^\top]}$$



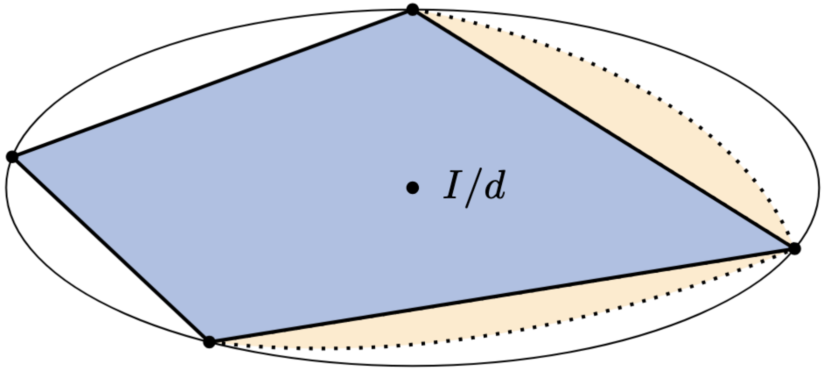
Are there any negative entries in the KD distribution?

Is this state a stabilizer state? (linear programming)



15% Extension of classical simulability!

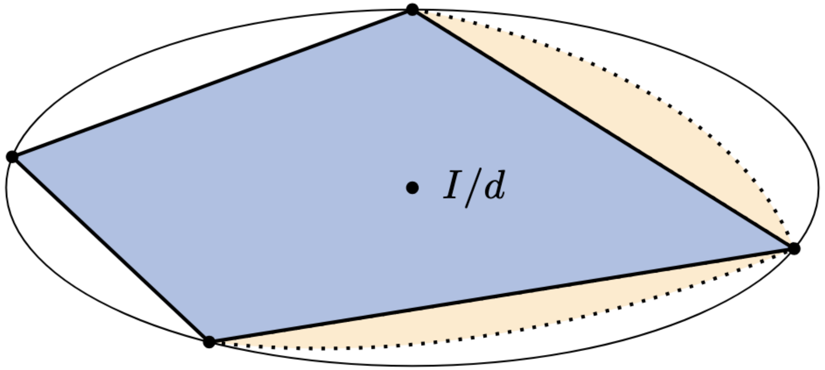
Conclusion:



- With the KD distribution, we can find the bound magic states
- By knowing more bound magic states, we can extend our classical simulability

Conclusion:





- With the KD distribution, we can find the bound magic states
- By knowing more bound magic states, we can extend our classical simulability



arXiv:2506.08092

Thank you!