Tensor algebra → particle Dynamics.

4-vectors.

ct x y z.

SR → spacetime is a pseudo-Euclidean manifold.

globally define Cartesian coordinate.

↓

Minkowski st.

ds^2 line element

$$ds^2 = \underbrace{\eta_{\mu\nu}}\, dx^\mu dx^\nu$$

$$\underbrace{\text{diag}}(+1,-1,-1,-1)$$

$$\begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

x^a a 1..N

$$\begin{matrix} (ct, & x, & y, & z) \\ \downarrow & \downarrow & \downarrow & \downarrow \\ x^0 & x^1 & x^2 & x^3 \end{matrix}$$

Note $\underbrace{\Gamma^\mu_{\nu\sigma}}_{\text{connection}} = 0.$ metric constant.

$$\eta_{\mu\nu} = \underbrace{\frac{\partial x'^\rho}{\partial x^\mu}} \frac{\partial x'^\sigma}{\partial x^\nu} \eta_{\rho\sigma}$$

$$\underbrace{x'^\mu = \Lambda^\mu{}_\nu x^\nu + \boxed{a^\mu}}_{=0}$$

→ (Poincaré) inhomogeneous Lorentz transf

$$\underline{\eta_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma}}$$

↓

$(\gamma \quad -\beta\gamma \quad 0 \quad 0 \quad$ x-direction.

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\gamma = (1-\beta^2)^{-\frac{1}{2}} \qquad \beta = \frac{v}{c}.$$

$$(\Lambda^{-1})^{\mu}{}_{\nu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}}$$

Proper lorentz transformation.

subgroup of full lorentz transf that.
only includes. transformation between
inertial frames with same spatial handedness
and exclude time reversal

$$[\det(\Lambda^{\mu}{}_{\nu})^2] = 1.$$

$$\mu = \nu = 0.$$

$$(\Lambda^{0}{}_{0})^2 = 1 + \sum_{i=1}^{3} (\Lambda^{i}{}_{0})^2 \geq 1$$

the subgroup of Proper LT [illegible].

$$\det(\Lambda^{\mu}{}_{\nu}) = 1 \qquad \Lambda^{0}{}_{0} \geq 1$$

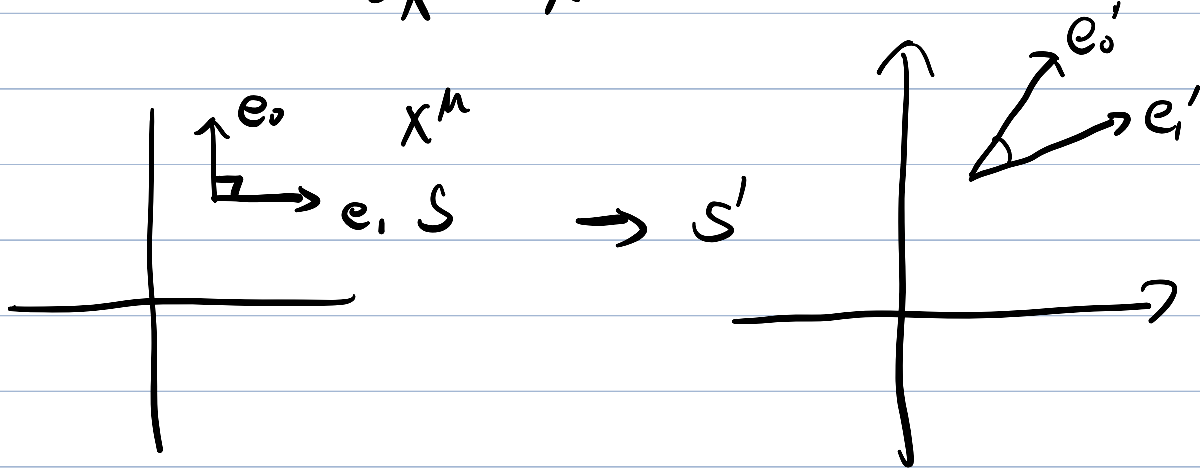
Cartesian basis vector.

$\frac{\partial}{\partial x^a}$. span the tangent space at any point.

$$e_\mu \equiv \frac{\partial}{\partial x^\mu}$$

$g(e_\mu, e_\nu) = \eta_{\mu\nu}$ → basis vectors are orthonormal.

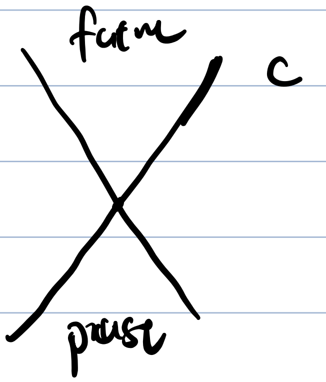
$$\frac{\partial}{\partial x'^a} = \frac{\partial x^b}{\partial x'^a}\frac{\partial}{\partial x^b}$$



$$e'_\mu = \Lambda_\mu^{\ \nu} e_\nu$$

↓

$$\frac{\partial}{\partial x'^\mu}$$



$\eta_{\mu\nu} v^\mu v^\nu > 0$ timelike

$\eta_{\mu\nu} v^\mu v^\nu < 0$ spacelike.

$=0$ null.

A massive particle follows a trajectory through spacetime $\to$ worldline.

proper time τ

$$ds^2 = c^2 d\tau^2.$$

4- vel

$$u^\mu = \frac{dx^\mu}{d\tau}$$

4-vel is future-pointing & time like.

$$\eta_{\mu\nu} u^\mu u^\nu = \left(\frac{ds}{d\tau}\right)^2 = c^2.$$

$$u^\mu = \left(c\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right)$$

$$= \frac{dt}{d\tau}\left(c, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$= \frac{dt}{d\tau}\left(c, \vec{u}\right)$$

$\vec{u}$ $\downarrow$ 3-vel.

$$c^2 = \eta_{\mu\nu} u^\mu u^\nu = \left(\frac{dt}{d\tau}\right)^2 (c^2 - |\vec{u}|^2)$$

$$\frac{dt}{d\tau} = \left(1 - \frac{|\vec{u}|^2}{c^2}\right)^{-\frac{1}{2}} = \gamma_u$$

Lorentz factor.

$$u'^\mu = \Lambda^\mu{}_\nu u^\nu \qquad \beta = \frac{v}{c}$$

$$\begin{pmatrix} \gamma_{u'} c \\ \gamma_{u'} u'^1 \\ \gamma_{u'} u'^2 \\ \gamma_{u'} u'^3 \end{pmatrix} = \begin{pmatrix} \gamma_v & -\beta\gamma_v & & 0 \\ -\beta\gamma_v & \gamma_v & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} \gamma_u c \\ \gamma_u u^1 \\ \gamma_u u^2 \\ \gamma_u u^3 \end{pmatrix}$$

first component.

$$\Rightarrow \frac{\gamma_u}{\gamma_{u'}} = \frac{1}{\gamma_v} \frac{1}{1 - \frac{u'v}{c^2}}$$

$$u'^1 = \frac{u^1 - v}{1 - u^1 v / c^2}$$

$$u'^2 = \ldots$$

$$u'^3 = \ldots$$

4- acceleration.

in inertial frame a free particle

has $\frac{d\dot{x}^i}{dt^2} = 0.$ $\vec{u} = \text{const}$ $\gamma_u = \text{const}.$

$$\frac{du^\mu}{d\tau} = 0. \Rightarrow \frac{Du^\mu}{D\tau} = 0.$$ intrinsic deriv.

along the worldline

free massive particles

move on timelike geodesics

in Minkowski space.

Relativistic mechanics of massive particle

4- mom. $p = mu$

massive particle.

$$p^\mu = (\gamma_u mc, \underbrace{\gamma_u m \vec{u}}_{\vec{p}})$$

span

the component, $\vec{P}$

3-mom

1. $\gamma_u \vec{u}$ $|\vec{u}| \ll c$ $\rightarrow$ $\vec{p} \approx m\vec{u}$

2. for a free particle. $\vec{p}$ constant since $\vec{u}$ is

3. Newton 2nd law takes form $\vec{f} = \frac{d\vec{p}}{dt}$.

$\vec{f}$ 3-force.

$E = \gamma_u mc^2$ energy = rest-mass E + kinetic en E

$\sim mc^2 + \frac{1}{2}mv^2$.

rate of working

$$\vec{u}\cdot\vec{f} = \vec{u}\cdot\frac{d\vec{p}}{dt}$$

$$= \vec{u}\cdot\frac{d}{dt}(\gamma_u m\vec{u})$$

$$= \gamma_u m\left(u\cdot a + \gamma_u^2\, \vec{u}\cdot\vec{a}\,\frac{|u|^2}{c^2}\right)$$

$$= \gamma_u^3 m\vec{u}\cdot\vec{a}$$

$$= mc^2 \frac{d\gamma_u}{}$$

$$= \frac{dE}{dt}.$$

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right)$$

$\Rightarrow$ energy-momentum invariant.

$$E^2 - |\vec{p}|^2 c^2 = m^2 c^4.$$

Force 4-vector.

acted on by a force, the 4-mom $\neq$ constant

4-force.

$$\frac{Dp^\mu}{D\tau} = f^\mu \Rightarrow g(f, u) = 0.$$

$\Rightarrow$ 4-vel and 4-force are necessarily orthogonal.

$$\frac{dE}{dt} = f \cdot u$$

$$f^\mu = \gamma_u \underbrace{\frac{d}{dt}\left(\frac{E}{c}, \vec{p}\right)}_{p} = \gamma_u\left(\frac{f \cdot u}{c}, \underbrace{\vec{f}}_{\text{3-force}}\right)$$

Photon.

$$E = |\vec{p}|c.$$

$$g(p,p) = 0.$$

null vector / four-poiny.