Energy - momentum tensor. (ideal fluids)

$T^{\mu\nu}$

T^{ij}

ρ $[\rho_0]$

rest-frame energy density.

$$T^{\mu\nu} = \mathrm{diag}(\rho c^2, p, p, p)$$

isotropic pressure.

$$T^{\mu\nu} = \underbrace{(\rho + p/c^2) u^\mu u^\nu - p g^{\mu\nu}}_{\text{4-vel}}$$

① instantaneous rest frame

$$u^\mu = (c, \vec{0}) \quad g^{\mu\nu} = \eta^{\mu\nu}$$

② $p \ll \rho c^2$ $\quad T^{\mu\nu}_{\text{ideal fluid}} \approx T^{\mu\nu}_{\text{dust}}$.

Conservation of energy & momentum.

$$\nabla_\mu T^{\mu\nu} = 0.$$

$\nabla_\mu (\ ) = 0.$ → conservation.

electromagnetic charge conservation.

$$\nabla_\mu j^\mu = 0.$$

local-inertial coordinates at P,

$0:$
$$\frac{\partial T^{00}}{\partial t} + c \sum \frac{\partial T^{i0}}{\partial x^i} = 0. \quad (1)$$

$i:$
$$\frac{\partial (cT^{i0}/c)}{\partial t} + \sum \frac{\partial T^{ij}}{\partial x^j} = 0. \quad (2)$$

(1) $\frac{\partial}{\partial t}$ (energy density) + $\vec{\nabla}\cdot$ (energy flux) $= 0.$

⇒ conservation of energy

continuity equation.

② $\frac{\partial}{\partial t}$ (momentum densy) $+ \vec{\nabla} \cdot$ (momem flux) $= 0$.

$\Rightarrow$ Conserny of momemy.

Example. ideal fluid.

$$T^{\mu\nu} = (\rho + \frac{p}{c^2}) u^\mu u^\nu - p g^{\mu\nu}$$

$$\boxed{\nabla_\mu T^{\mu\nu}.} \qquad \nu$$

$$\nabla_\mu \left[(\rho + p/c^2) u^\mu u^\nu - p g^{\mu\nu} \right] = 0.$$

$$\left\{ \begin{aligned} & u^\nu u^\mu \nabla_\mu (\rho + p/c^2) + (\rho + p/c^2)(\nabla_\mu u^\mu) u^\nu \\ & + \underline{(\rho + p/c^2) u^\mu \nabla_\mu u^\nu} - \boxed{\nabla_\mu p\, g^{\mu\nu} - p \boxed{\nabla_\mu g^{\mu\nu}}} = 0 \end{aligned} \right.$$

$$\underline{g^{\mu\nu} \nabla_\mu p}$$

$$\nabla^\nu p$$

parallel & perpendicular to u^ν.

① contract with u_ν (parallel)

$$c^2 u^\mu \nabla_\mu (\rho + P/c^2) + c^2(\rho + P/c^2)(\nabla_\mu u^\mu)$$
$$- u^\mu \nabla_\mu P = 0.$$

$$(u^\mu \nabla_\mu u^\nu)\, u_\nu = 0.$$

$$\Rightarrow \quad g_{\nu\rho} u^\nu u^\rho = c^2.$$

$$u^\mu \nabla_\mu (g_{\nu\rho} u^\nu u^\rho) = 0.$$

$$\Rightarrow \quad g_{\nu\rho} (u^\mu \nabla_\mu u^\nu)\, u^\rho$$
$$+ g_{\nu\rho} u^\nu (u^\mu \nabla_\mu u^\rho) = 0.$$

$$= 2 u_\nu (u^\mu \nabla_\mu u^\nu) = 0.$$

②. perpendicular.

$$(\rho + \frac{p}{c^2}) u^\mu \nabla_\mu u^\nu = (g^{\mu\nu} - \frac{u^\mu u^\nu}{c^2}) \nabla_\mu p$$

Local-inertial frame. at P.

4-vel

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}(c, \frac{dx^i}{dt}) = \underset{\sim}{\frac{dt}{d\tau}}(c, \vec{u}^i)$$

$$g_{\mu\nu} u^\mu u^\nu = c^2.$$

Consider Newtonian limit $|u^i| \ll c$.

$$\frac{dt}{d\tau} \sim 1$$

$$u^\mu \approx (c, \vec{u}^i)$$

speed of all particle in rest-frame. $\ll c$

$$p \ll \rho c^2.$$

$$\underset{\sim}{\nabla_\mu (\rho u^\mu)} + \frac{p}{c^2} \nabla_\mu u^\mu = 0.$$

(metric connection $= 0$ (at P)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^i}(\rho u^i) \approx 0.$$

Newtonian continuity eqn.

$$\rho u^\mu \frac{\partial u^\nu}{\partial x^\mu} = (\eta^{\mu\nu} - \frac{u^\mu u^\nu}{c^2})\frac{\partial p}{\partial x^\mu}$$

$\nu = 0.$

$$\rho u^\mu \frac{\partial u^0}{\partial x^\mu} = (\eta^{\mu 0} - \frac{u^\mu u^0}{c^2})\frac{\partial p}{\partial x^\mu}$$

$\boxed{\begin{matrix} p \ll \rho c^2 \\ u \ll c. \end{matrix}}$ $\frac{p}{\rho c^2}$ $u/c. \ll 1.$

$\nu = i$

$$\cancel{\rho c}\left(\frac{\partial u^i}{\partial t} + \sum u^j \frac{\partial u^i}{\partial x^j}\right) \simeq -\sum \delta^{ij} \frac{\partial p}{\partial x^j} - \frac{u^i}{c^2}\frac{\partial p}{\partial t}$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u} \simeq -\frac{\vec{\nabla} p}{\rho}$$

Euler eqn. (for ideal fluid)

Newtonian fluid mechanics.

Recall in Newtonian Gravity.

$$\nabla^2 \Phi = 4\pi G \rho$$

weak-field limit.

$$g_{00} \approx (1+\frac{2\Phi}{c^2})$$

slow-moving fluid $T_{00} \approx \rho c^2$

(localinertial coord)

the Poisson eqn in the limit of weak fields and non-relativistic speeds is

$$\nabla^2 g_{00} = \frac{8\pi G}{c^4} T_{00} \to \text{energy momen.}$$

curvature of spacetime.

$$K_{\mu\nu} = \kappa T_{\mu\nu}$$

$$\kappa = \frac{8\pi G}{c^4}$$

$$\nabla_\mu T^{\mu\nu} = 0.$$

$$\nabla^\mu \underline{K_{\mu\nu}} = 0.$$

$$\nabla^\mu (\qquad) = 0.$$

$$\downarrow$$

$$\underline{\nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0.}$$

Bianchi identy.

$$\begin{cases} G_{\mu\nu} = R_{\mu\nu} - \dfrac{g_{\mu\nu}}{2} R = -\kappa T_{\mu\nu} \\ \nabla^\mu G_{\mu\nu} = 0 \end{cases}$$

Einstein field eqn.

in Empty space. (in vaccum)

$T = 0.$

$$R_{\mu\nu} = \frac{g_{\mu\nu}}{2} R.$$

Weak-field limit (Recover)

$$R_{00} = -\kappa\left(T_{00} - \tfrac{1}{2} g_{00} T\right)$$

$p \ll \rho c^2$.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$T_{\mu\nu} \approx \rho u_\mu u_\nu$$

$$T \approx g^{\mu\nu} \rho\, u_\mu u_\nu = \rho c^2.$$

$$u^\mu \approx (c, u^i)$$

$$u_0 = g_{0\mu} u^\mu \approx g_{00} c \approx c.$$

$$T_{00} \approx \rho u_0 u_0 \approx \rho c^2.$$

$$R_{00} \approx -\tfrac{1}{2} \kappa \rho c^2$$

$$R_{00} = -\partial_\mu \Gamma^\mu_{00} + \partial_0 \Gamma^\mu_{\mu 0} + \Gamma^\nu_{\mu 0} \Gamma^\mu_{0\nu}$$

$$- \Gamma^\nu \Gamma^\mu$$

$$\Gamma_{00} \Gamma_{\mu\nu}$$

$$\approx -\sum_i \frac{\partial \Gamma^i_{00}}{\partial x^i}$$

$$\Gamma^i_{00} \approx \frac{1}{2} \frac{\partial h_{00}}{\partial x^i}$$

$$R_{00} \approx -\frac{1}{2} \vec{\nabla}^2 h_{00}$$

$$\Downarrow$$

$$\nabla^2 h_{00} \approx \frac{8\pi G}{c^2} \rho$$

$$\Downarrow$$

$$g_{00} \approx 1 + \frac{2\Phi}{c^2} \qquad h_{00} \approx \frac{2\Phi}{c^2}.$$

$$\approx \eta + h.$$

$$\nabla^2 \Phi = 4\pi G \rho.$$

Cosmological constant.

$$\underline{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu}}$$

div free.

2. constructed from metric and its first two derivatives.

3. linear in second derivatives of metric

Lovelock's theorem.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \underline{\Lambda} g_{\mu\nu} = -\kappa T_{\mu\nu}.$$

$$\nabla_\rho g_{\mu\nu} = 0.$$