Vector & Tensor algebra.

(pseudo) Riemannian manifold

physical law → reduce locally-inertial coordinate.

1. Scalar fields.

A real (complex) scalar field defined on a (subset) manifold $\mathcal{M}$ assign real (complex) number. to each point P in (subset) $\mathcal{M}$.

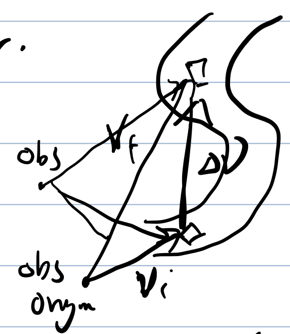
⇓

$\mathcal{M}$ x^1 x^2 ..... x^N.

$x^a \in$

$\phi(x^a)$ -> number.

$\underline{\phi(x^a) = \phi(x'^a)}$



2. Vector fields

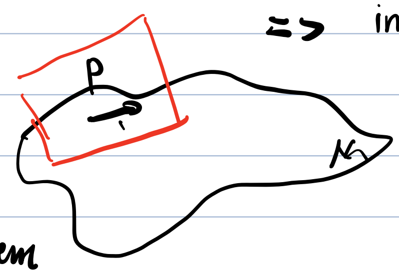
displacement vec. connect two points in space.

local vector. measured at a given observation point and solely to that point.

Displacemen. => M has an embedding in some higher-dimensional Euclidean space.

=> intrinsic geometry.

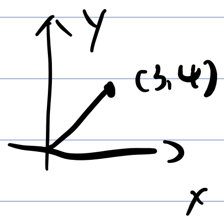
T_p



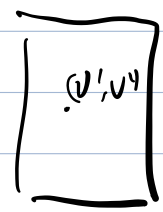
tangent space at P.

T_p => at P consider operator $\underline{v} = v^a \underbrace{\frac{\partial}{\partial x^a}}$

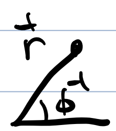
x^a coordinate chart.



T_p



$$3\vec{e}_x + 4\vec{e}_y \qquad v = v^1 \frac{\partial}{\partial x^1} + v^2 \frac{\partial}{\partial x^2}$$



x'

$$\frac{\partial}{\partial x'^a} = \frac{\partial x^b}{\partial x'^a} \frac{\partial}{\partial x^b}$$

$$v'^a = \frac{\partial x'^a}{\partial x^b} v^b$$

$$v \to v'^a \frac{\partial}{\partial x'^a} = \frac{\partial x'^a}{\partial x^b} v^b \frac{\partial x^c}{\partial x'^a} \frac{\partial}{\partial x^c}$$

$$= v^b \frac{\partial}{\partial x^c} \delta^c_b$$

$$= v^b \frac{\partial}{\partial x^b} = v$$

3. Dual vector fields.

consider gradient of a scalar field.

ϕ. $\quad \frac{\partial \phi}{\partial x^a} = X_a$

$$X_a' = \frac{\partial \phi'}{\partial x'^a} = \frac{\partial x^b}{\partial x'^a} \frac{\partial \phi}{\partial x^b} = \frac{\partial x^b}{\partial x'^a} X_b$$

define $\underbrace{X_a' = \frac{\partial x^b}{\partial x'^a}}$ X_b under a coordinate transf

as components of a dual vector.

(consider T_p at p)

(T_p^*)

dual

$$\underset{\text{vector}}{X_a' v'^a} = \frac{\partial x^b}{\partial x'^a} X_b \frac{\partial x'^a}{\partial x^c} v^c = X_b v^c \delta^b_c$$

$$= X_b v^b$$

=> invariant.

4. Tensor fields.

if $T_p(M)$ & $T_p^*(M)$ take k dual vectors and l vectors at p and returns a number.

=> we say such a tensor to be of type (k, l) and have a rank $\underline{k+l}$.

T_{ab} type $(0,2)$ $0/k$ -> contravariant

T^{ab} type $(2,0)$ $0/l$ -> covariant.

$$T'^{a \ldots b}{}_{c \ldots d} = T^{p \ldots q}{}_{r \ldots s} \frac{\partial x'^a}{\partial x^p} \cdots \frac{\partial x'^b}{\partial x^q} \frac{\partial x^r}{\partial x'^c} \cdots \frac{\partial x^s}{\partial x'^d}$$

$k+l=0$. rank $=0$. => scalar fields.

$k=1$, $l=0$. => vectors.

$k=0$, $l=1$ => dual vector.

4.1 contraction.

=> setting an upstairs and downstairs index equal and summing.

$(k,l) \rightarrow (k-1, l-1)$

$$\boxed{S^a \equiv T^{ab}{}_b}$$

$$T'^{ab}{}_c = \frac{\partial x'^a}{\partial x^p} \frac{\partial x'^b}{\partial x^q} \frac{\partial x^r}{\partial x'^c} T^{pq}{}_r$$

V_a

V^a

ϕ ~~0~~.

$$S'^a = \frac{\partial x'^a}{\partial x^p} \frac{\partial x'^b}{\partial x^q} \frac{\partial x^r}{\partial x'^b} T^{pq}{}_r$$

$$= \frac{\partial x'^a}{\partial x^p} \delta^r_q T^{pq}{}_r \longrightarrow T^{pq}{}_q$$

$$= \frac{\partial x'^a}{\partial x^p} S^p$$

5. Metric Tensor.

$$ds^2 = \underbrace{g_{ab}}_{\text{metric}} dx^a dx^b \qquad \text{line element.}$$

$$g'_{ab} = \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} g_{cd}$$

type $\underline{(0,2)}$ tensor.

$\Rightarrow$ metric tensor

$$\underbrace{g(u,v)}_{\text{scalar.}} = g_{ab}\, u^a\, v^b.$$

Dual. vec

this metric provides a map between vec & dual vec at a point between $T_p(M)$ & $T_p^*(M)$

$$v_a \equiv g_{ab} v^b.$$

$$\underline{T_{ab}} \equiv \underline{g_{ac} T^c{}_b}$$

$$T_{abc} \equiv g_{ap} g_{bq} T^{pq}{}_c$$

$$g_{a\cdots b}$$

5.1 inverse. metric.

type (2,0).

$$\underline{(g^{-1})^{ab} g_{bc}} = \underline{\delta^a_c}$$

$$(g'^{-1})^{ab} = \frac{\partial x'^a}{\partial x^c} \frac{\partial x'^b}{\partial x^d} (g^{-1})^{cd}$$

$$(g'^{-1})^{ab} g'_{bc} = \delta^a_c \Rightarrow (g^{-1})^{ab}$$

$$T_a{}^{bc} \equiv g_{ad} g^{ce} T^{db}{}_e$$

6. Tensor (vector) calculus on Manifolds

laws of physics ⟶ how to do derivative on a general manifold with Tensor?

$f(x, y)$ $\quad \frac{df}{dt}$.

7. Derivatives of a scalar field.

$y = 2x$

$\frac{dy}{dx} = 2.$

$$\frac{\partial \phi'}{\partial x'^a} = \frac{\partial x^b}{\partial x'^a} \frac{\partial \phi}{\partial x^b} \quad (f)$$

$$\delta \phi = \frac{\partial \phi}{\partial x^a} \delta x^a$$

vector

coordinate separation δx^a.

8. Tensor fields.

$$\frac{\partial v'^b}{\partial x'^a} = \frac{\partial}{\partial x'^a} \left(\frac{\partial x'^b}{\partial x^c} v^c \right)$$

$\partial x^d \quad \partial \quad \ldots v'^b$

$$= \frac{\partial x^d}{\partial x'^a} \frac{\partial}{\partial x^d} \left(\underset{\sim\sim}{\frac{\partial x'^b}{\partial x^c}} v^c \right)$$

$$= \underbrace{\frac{\partial x^d}{\partial x'^a} \frac{\partial v^c}{\partial x^d} \frac{\partial x'^b}{\partial x^c}}_{\text{type (1,1) transf.}} + \frac{\partial x^d}{\partial x'^a} v^c \underset{\sim\sim}{\frac{\partial^2 x'^b}{\partial x^d \partial x^c}}$$

introduce covariant derivative.

the covariant derivative of a type (k, l) tensor $T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$ is a type $(k, l+1)$ tensor, denoted by

$$\underset{\sim}{\nabla_c} T^{a_1 \dots a_k}{}_{b_1 \dots b_l}.$$

Properties:

(1) scalar fields. $\nabla_a \phi = \frac{\partial \phi}{\partial x^a}$.

(2) Linearity $\nabla_c (\alpha T^{a_1 \dots a_k}{}_{\dots} + \beta S^{a_1 \dots a_k}$

$$= \alpha \nabla_c T + \beta \nabla_c S.$$

(3) Leibnitz $\nabla_f (T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} S^{c_1 \cdots c_m}{}_{d_1 \cdots d_n})$

$$= (\nabla_f T) S + T (\nabla_f S)$$

$$\boxed{\nabla_a v^b} = \frac{\partial v^b}{\partial x^a} + \Gamma^b_{ac} v^c$$

type (1,1) tensor.

connection coefficient.

$$\nabla'_a v'^b = \frac{\partial v'^b}{\partial x'^a} + \Gamma'^b_{ac} v'^c$$

$$= \frac{\partial x^d}{\partial x'^a} \frac{\partial x'^b}{\partial x^c} \frac{\partial v^c}{\partial x^d} + \frac{\partial x^d}{\partial x'^a} \frac{\partial^2 x'^b}{\partial x^d \partial x^c} v^c + \Gamma'^b_{ac} \frac{\partial x'^c}{\partial x^d} v^d$$

$$= \frac{\partial x^d}{\partial x'^a} \frac{\partial x'^b}{\partial x^c} \nabla_d v^c - \frac{\partial x^d}{\partial x'^a} \frac{\partial x'^b}{\partial x^c} \Gamma^c_{de} v^e$$

$$+ \frac{\partial x'^a}{\partial x'^a} \frac{\partial^2 x'^b}{\partial x^d \partial x^c} V^c + \Gamma'^b_{ac} \frac{\partial x'^c}{\partial x^d} V^d.$$

$$\underset{\sim}{\Gamma'^a_{bc}} = \frac{\partial x'^a}{\partial x^d} \frac{\partial x^e}{\partial x'^b} \frac{\partial x^f}{\partial x'^c} \Gamma^d_{ef} - \underbrace{\frac{\partial x^d}{\partial x'^b} \frac{\partial x^e}{\partial x'^c}}_{} \frac{\partial^2 x'^a}{\partial x^d \partial x^e}$$

DO NOT transf as a inhomogenary tensor.

$\Rightarrow \Gamma^a_{bc}$ is unique up to a ~~type~~ (1, 2) tensor.

$$\nabla_a (u^b V^c) = (\underset{\sim}{\nabla_a u^b}) V^c + u^b (\underset{\sim}{\nabla_a V^c})$$

$$= \left(\frac{\partial u^b}{\partial x^a} + \Gamma^b_{ad} u^d\right) V^c$$

$$+ u^b \left(\frac{\partial V^c}{\partial x^a} + \Gamma^c_{ad} V^d\right)$$

$$= \frac{\partial}{\partial x^a}(u^b V^c) + \Gamma^b_{ad} u^d V^c + \Gamma^c_{ad} u^b V^d$$

$$\nabla_a (X_b V^b) = \frac{\partial X_b}{\partial x^a} V^b + X_b \frac{\partial V^b}{\partial x^a}$$

$$\nabla_a (\Lambda_b V^b) = \frac{\partial \Lambda_b}{\partial x^a} V^b + \Lambda_b \frac{\partial V^b}{\partial x^a}.$$

Note,

metric tensor.

$$\nabla_c g^a{}_b = \cancel{\partial_c \delta^a_b} + \underline{\Gamma^a_{cd}\delta^d_b} - \Gamma^d_{cb}\delta^a_d$$

$$= \Gamma^a_{cb} - \Gamma^a_{cb} = 0.$$

$\Rightarrow$ covariant derivative commutes with contraction. $(g^a{}_b = \delta^a_b)$

$$\nabla_c \delta^a_b = 0.$$

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc})$$

$\Rightarrow$ allows computation of the connection coeff in an arbitrary coord sys

$$\nabla_a g^{bc} = 0. \qquad g_{ab} g^{bc} = \delta^c_a.$$

$$g = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & \text{[illegible]} & 1 \end{pmatrix}$$

$$(\det M)^{-1} \partial_c \det M$$

$$= \mathrm{Tr}(M^{-1} \partial_c M)$$

tr (matrix)

$\mathrm{tr}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3.$

$\cdot \ \ln(\det M) = \mathrm{Tr}(\ln M)$

$$\Gamma^{a}_{ac} = \frac{1}{2} g^{-1} \partial_c g = |g|^{-1/2} \partial_c |g|^{\frac{1}{2}}$$

9. Div, curl, Laplacian.

$$\nabla_a V^a = \partial_a V^a \pm \Gamma^{a}_{ab} V^b$$

$$= |g|^{-\frac{1}{2}} \partial_a \left(|g|^{\frac{1}{2}} V^a \right)$$

$$(\mathrm{curl}\, X)_{ab} = \nabla_a X_b - \nabla_b X_a$$

$(\nabla \times V.)$

$$= \partial_a X_b - \cancel{\Gamma^{c}_{ab} X_c}$$
$$- \left(\partial_b X_a - \cancel{\Gamma^{c}_{ba} X_c} \right)$$

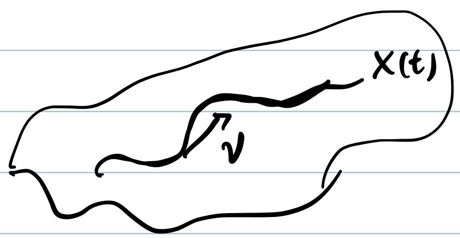
$$= \partial_a X_b - \partial_b X_a$$

$$\nabla^2 \phi \equiv \nabla_a \left(g^{ab} \nabla_b \phi \right)$$

$$= |g|^{-\frac{1}{2}} \partial_a \left(|g|^{\frac{1}{2}} g^{ab} \partial_b \phi \right)$$

$$\nabla^2 T^{ab} = g^{cd} \nabla_c \nabla_d T^{ab}$$

10. Intrinsic derivative.

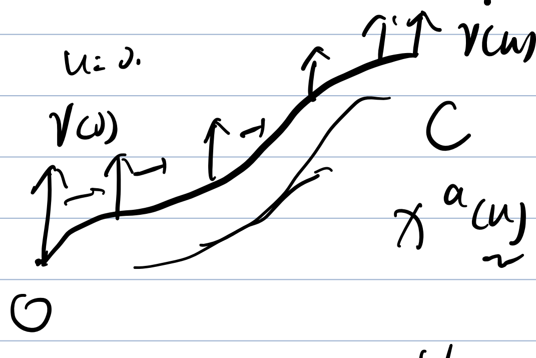


$$\frac{Dv^a}{Du} = \frac{dx^b}{du} \nabla_b v^a$$

$$= \frac{dx^b}{du} \left(\partial_b v^a + \Gamma^a_{bc} v^c \right)$$

$$= \underbrace{\frac{dv^a}{du}}_{\text{usual ordinary}} + \frac{dx^b}{du} \Gamma^a_{bc} v^c .$$

11. Parallel transport.



length / direction of v are conserved.

the resulting vector field $v(u)$ is said to be parallel transport

$$\frac{Dv^a}{Du} = 0.$$

$$\frac{DT^{ab}}{Du} = 0.$$

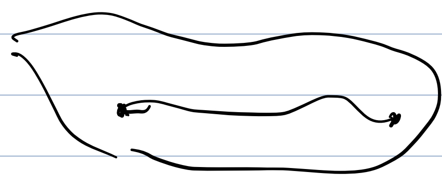
Length $$\frac{d|v|^2}{du} = \frac{D}{Du}(g_{ab}v^a v^b)$$

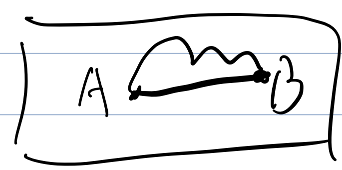
$$= 2g_{ab}v^a \frac{Dv^b}{Du} = 0.$$

12. Geodesic curves.

=> straight lines on a Euclidean space.

free particles in GR follow Geodesic curve in space time.





tangent vector $t^a = \frac{dx^a}{}$

du.

$g(t,t) > 0$ timelike.

$g(t,t) < 0$ spacelike.

$g(t,t) = 0.$ null (lightlike)

non-null curve.

$$|t| = \Big|\underbrace{g_{ab}\frac{dx^a}{du}\frac{dx^b}{du}}_{t^2}\Big|^{\frac{1}{2}} = \Big|\frac{ds}{du}\Big|$$

$s \to$ path length

Relation to parallel transport.

$$\frac{Dt^a}{Du} = 0.$$

non-null geodesic $\Leftrightarrow$ tangent vector.

Particle Dynamics.

tensor algebra / calculus.

$\Rightarrow$ SR (4-vector)

rotating frame

Minkowski spacetime.

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

$$\downarrow$$

$$\mathrm{diag}(+1, -1, -1, -1)$$

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

g_{ab} g^{ab} $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$

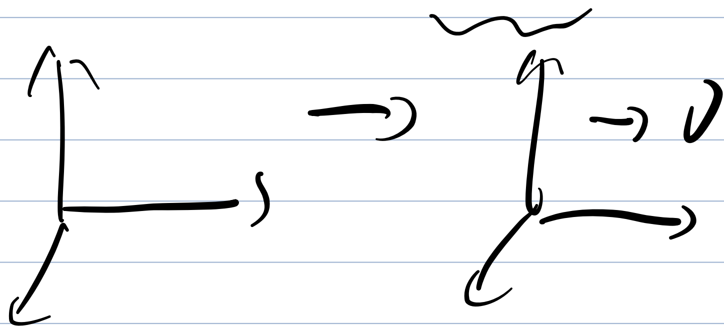
Lorentz transfm

$$\eta_{\mu\nu} = \frac{\partial x^\rho}{} \frac{\partial x^\sigma}{} \eta_{\rho\sigma}$$

$$\partial x'^{\mu} \qquad \partial x'^{\nu}$$

$$\downarrow \qquad \downarrow$$

$$\Lambda^{\rho}{}_{\mu} \quad \Lambda^{\sigma}{}_{\nu}$$



$$x' = \gamma (x - ct\beta)$$

$$\uparrow$$

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$