Riemann curvature tensor.

$$\boxed{\underline{R_{abc}{}^{d}} = -\partial_a \Gamma^d_{bc} + \partial_b \Gamma^d_{ac} + \Gamma^e_{ac}\Gamma^d_{be} - \Gamma^e_{bc}\Gamma^d_{ae}.}$$

(1,3)

$$\nabla_a \nabla_b V_c - \nabla_b \nabla_a V_c = R_{abc}{}^{d} V_d$$

$$\nabla_b \nabla_a V_c - \nabla_a \nabla_b V_c = -R_{abc}{}^{d} V_d$$

$$= R_{bac}{}^{d} V_d$$

antisym (first 2 indices)

① $R_{abc}{}^{d} = -R_{bac}{}^{d}$

② $R_{abc}{}^{d} + R_{cab}{}^{d} + R_{bca}{}^{d} = 0.$

cyclic sym (first 3 indices)

③ R_{abcd} (0,4) → other sym.

Local Cartesian coordinate p

$$(R_{abcd})_p = -(g_{de}\partial_a \Gamma^e_{bc} - g_{de}\partial_b \Gamma^e_{ac})_p$$

$$\Gamma^e_{bc} = \frac{1}{2} g^{ef} (\partial_b g_{cf} + \partial_c g_{bf} - \partial_f g_{bc})$$

$$\left(g_{de}\,\partial_a \Gamma^e_{bc}\right)_p = \tfrac{1}{2}\left(\partial_a\partial_b g_{cd} + \partial_a\partial_c g_{bd} - \partial_a\partial_d g_{bc}\right)_p$$

$$\Rightarrow (R_{abcd})_p = \tfrac{1}{2}\left(\partial_a\partial_d g_{bc} + \partial_b\partial_c g_{ad} - \partial_a\partial_c g_{bd} - \partial_b\partial_d g_{ac}\right)_p$$

$\rightarrow$ $R_{abcd} = -R_{abdc}$ (last 2 index)

$\rightarrow$ $R_{abcd} = R_{cdab}$

$g = \begin{pmatrix} \quad \end{pmatrix}$

$R = \begin{pmatrix} 0 & 0 & 0 & 1 \\ & 0 & 0 & 1 & . \end{pmatrix}$

1D curvature tensor vanishes.

$R_{1111} \rightarrow 0.$

2D. antisym $\rightarrow R_{1212}$

3D R has 6 independent components.

- R_{abcd} $\quad 3$

$\boxed{12}$

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

ab = $\begin{array}{|c|} 13 \\ 23 \end{array}\Big\} 3$

3x3 sym

$3 \times 3 = 9.$

$3 \times 3 - 3 = 6.$

$R_{[abc]d} = 0.$ true → trivial.

$a = c$ or
$a = b$ or
$b = c$ or.

4D → 20 indep components.

$$R_{012\boxed{3}} + R_{120\boxed{3}} + R_{201\boxed{3}} = 0.$$

ND → $\dfrac{N^2(N^2-1)}{12}$

Bianchi identity.

$$\nabla_a R_{bcd}{}^e + \nabla_b R_{cad}{}^e + \nabla_c R_{abd}{}^e = 0.$$

⇒ tensor identity. (involves covariant derivative)

$$\nabla_{[a} R_{bc]d}{}^e = 0.$$

simple proof.

local Cartesian coordin. on P.

$$(\nabla_{\underline{a}} R_{\underline{bc}d}{}^{e})_p = \Big(\partial_a[-\partial_b\Gamma^e_{cd} + \partial_c\Gamma^e_{bd} + \Gamma^f_{bd}\Gamma^e_{cf} - \Gamma^f_{cd}\Gamma^e_{bf}]\Big)_p$$

$$= \big(-\partial_a\partial_b\Gamma^e_{cd} + \partial_a\partial_c\Gamma^e_{bd}\big)_p.$$

Ricci tensor.

Lower-rank tensor $\rightarrow$ contraction.

$$R_{abcd} = R_{[ab]cd} = R_{ab[cd]}$$

the only option for contraction is

across 1^{st} & 2^{nd} pair

Ricci tensor.

$$R_{ab} \equiv R_{cab}{}^{c}$$

$$\delta^c_d\big(R_{abc}{}^{d} + R_{cab}{}^{d} + R_{bca}{}^{d}\big)$$

$$= R_{ab} - R_{ba} = 0.$$

Ricci tensor → symmetric.

Ricci Scalar. (or curvature scalar)

$$R \equiv g^{ab} R_{ab}.$$

if a manifold is flat in some region.
the curvature tensor will vanish
so will the Ricci Tensor & scalar

Contract the Bianchi identity.

$$0 = \delta^b_e \left(\nabla_a R_{bcd}{}^e + \nabla_c R_{abd}{}^e + \nabla_b R_{cad}{}^e \right)$$

$$= \nabla_a R_{cd} - \nabla_c R_{ad} + \nabla^b R_{cadb}$$

further contract..

$$0 = g^{ad} \left(\nabla_a R_{cd} - \nabla_c R_{ad} + \nabla^b R_{cadb} \right)$$

dummy var.

$$= \nabla^d R_{cd} - \nabla_c R + \nabla^b R_{cb} \to d$$

$$= 2\nabla^d R_{cd} - \nabla_c R.$$

Contracted Bianchi identity

$$\nabla^a \underbrace{\left(R_{ab} - \frac{1}{2} g_{ab} R \right)}_{} = 0.$$

divergence

ET is symmetric & div-free.

$$G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R.$$

Einstein tensor.

Gravitational field equations.

Einstein eqn $\Leftarrow$ $\begin{cases} \text{Einstein tensor. ✓} \\ \text{energy-mom tensor.} \end{cases}$

Newtonian gravity Poisson eqn $\begin{cases} \text{Laplacian} \\ \text{mass density dist.} \end{cases}$

GR

↳ find appropriate tensor.

generalise mass density to describe energy in spacetime.

$\Rightarrow$ 1. non-interacting particle. 2. rest mass m.
3. no vel dispersion.

↓

dust.

at event P $\Rightarrow$ all have same 4-vel

$u^{\mu}(x)$

$\Rightarrow$ energy density ρc^2

there exists a local-inertial frame.

number density n_0

$$\rho_0 c^2 = \underbrace{m n_0 c^2}.$$

rest frame

local-inertial S (3-vel $\vec{u}$)

number density $\gamma_u n_0$ (length contraction)

energy of each particle $\gamma_u m c^2$.

$$\rho c^2 = (\gamma_u n_0)\, \gamma_u m c^2 = \gamma_u^{2} \rho_0 c^2 .$$

it's not a Lorentz scalar. tensor (2,0)

$$T^{\mu\nu}(x) = \rho_0(x)\, u^{\mu}(x)\, u^{\nu}(x)$$

scalar 4-vector

$$\underset{\sim}{T^{00}} = \gamma_u^2 \rho_0 c^2 .$$

$$\underbrace{u^0 = \gamma_u c}$$

$$T^{i0} = \rho_0 u^i u^0 \qquad u^\mu = \gamma_u (c, \vec{u})$$

$$= m n_0 (\gamma_u u^i)(\gamma_u c)$$

$$= c \underbrace{(\gamma_u n_0)}_{\text{num density}} \underbrace{(m \gamma_u u^i)}_{\text{3-momentum}}$$

$\Rightarrow$ momentum density $(\times c)$

energy flux in i-th direction..

= energy density × partial 3-vel.

$$= \gamma^2 n_0 m c^2 \qquad u^i$$

$$= c T^{i0}.$$

$$T^{ij} = \rho_0 u^i u^j$$

$$= m n_0 (\gamma_u u^i)(\gamma_u u^j)$$

$$= (\gamma_u^2 m n_0 u^i) u^j$$

i-th component of 3-mom dens

× j-th component of 3-vel.

flux of i-th component of 3-mom

along j-th direct.

$T^{\mu\nu}$ applies to other sources as well

e.g. electromagnetic

gas → heat conduction / bulk motion

must consider. velocity dispersion. / shear stress

=> ideal fluids.

$$T^{\mu\nu} = \mathrm{diag}(\underbrace{\rho c^2}_{\text{rest frame}}, p, p, p)$$

→ isotropic

ery deny.

pressure.