1. Concept of Manifold.

The spacetime of SR - Minkowski spacetime is an example of Manifold.

Informally, an N-dimentional manifold is a set of object that locally resembles N D Euclidean space $\mathbb{R}^N$

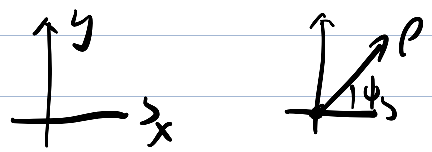
$\Downarrow$

there exists a map ϕ from ND manifold M to an open subset of $\mathbb{R}^N$ that is one-to-one and onto

x^a $\quad \mathbb{R}^N$ $\quad x^1, x^2 \cdots x^N$

$a = 1 \ldots N$

2. Coordinate.



The coordinate are not unique. (ϕ map changes)

E.g. 1

The Euclidean plane $\mathbb{R}^2$ is a 2D manifold that can be covered globally with the usual cartesian coordinate x, y.

(ρ, ϕ) $\quad 0 \le \rho \le \infty \quad 0 \le \phi < 2\pi$

$\rho = 0$. indeterminare.

e.g 2. coordinate (θ, ϕ) on the 2-phere S^2

$x^2 + y^2 + z^2 = 1$

$0 \le \theta \le \pi \quad 0 \le \phi < 2\pi$ $\quad \theta = 0$, $\theta = \pi$

$\phi \Rightarrow$ indeterminare.

curve. / plane (surface)

N D $\quad M < N \Rightarrow$ submanifold. (surface)

$x^a = x^a(u)$

$\hookrightarrow u^1 \; u^2 \dots u^N$

$\downarrow$

$x^a = x^a(u^1, u^2 \dots u^M)$

$M = N-1 \Rightarrow$ hypersurfen.

$f(x^1, x^2, \dots x^N) = 0.$

Coordinate transformation.

$x^a \Longleftrightarrow x'^a$

consider two neighborng points P & Q.

with coordinate x^a, $x^a + dx^a$.

$$dx'^a = \sum_{b=1}^{N} \frac{\partial x'^a}{\partial x^b} dx^b$$

$$\widetilde{J}\text{(acobian)} = \begin{pmatrix} \frac{\partial x'^1}{\partial x^1} & \cdots & \frac{\partial x'^1}{\partial x^N} \\ \vdots & & \\ \frac{\partial x'^N}{\partial x^1} & \cdots & \frac{\partial x'^N}{\partial x^N} \end{pmatrix}$$

$$\underbrace{\sum_{b=1}^{N}}\ \frac{\partial x'^a}{\cancel{\partial x^b}} \frac{\cancel{\partial x^b}}{\partial x'^c} = \frac{\partial x'^a}{\partial x'^c} = \delta^a_c$$

$$J \quad 1/J \quad = \quad 1$$

→ Einstein summation convention.

index occurs twice ↗ subscript ↓ ↘ superscript ↑ ⇒ sum over the Σ 1 to N.

$$dx'^a = \frac{\partial x'^a}{\partial x^b} dx^b \quad \frac{dx^a}{\partial x^b}$$

3. Local geometry of Riemannian manifolds.

$ds^2 = \underline{g_{ab}}(x)\, dx^a dx^b$. quadratics of the coordinate differentials.

$$\Delta s^2 = \int_A^B ds^2$$

$ds^2 > 0$ Riemannian.

$ds^2 \lessgtr 0$. pseudo-Riemannian. η

metric

g_{ab} ⇒ chosen to be symmetric.

$$g_{ab} = \frac{1}{2}(g_{ab}+g_{ba}) + \underbrace{\frac{1}{2}(g_{ab}-g_{ba})}$$

$$(g_{ab}-g_{ba})dx^a dx^b = \underbrace{g_{ab}\, dx^a dx^b - g_{ba}\, dx^b dx^a}$$

$$= 0.$$

$$g'_{cd}(x') = g_{ab}(x(x'))\frac{\partial x^a}{\partial x'^c}\frac{\partial x^b}{\partial x'^d}$$

Exampl 1. 2-sphere in $\mathbb{R}^3$

Consider the 2-phere embedded in $\mathbb{R}^3$, the embeding space has $ds^2 = dx^2 + dy^2 + dz^2$

$$x^2+y^2+z^2 = a^2. \qquad (1)$$

$$2x\,dx + 2y\,dy + 2z\,dz = 0.$$

$$dz = -\frac{(x\,dx+y\,dy)}{z} = \frac{-(x\,dx+y\,dy)}{\sqrt{a^2-x^2-y^2}}$$

$$ds^2 = dx^2+dy^2 + \frac{(x\,dx+y\,dy)^2}{a^2-(x^2+y^2)}$$

$$x = \rho\cos\phi \qquad y = \rho\sin\phi.$$

$$x\,dx + y\,dy = \rho\, d\rho$$

$$ds^2 = \frac{a^2 d\rho^2}{a^2-\rho^2} + \rho^2 d\phi^2$$

e.g. 3-sphere in $\mathbb{R}^4$.

$$x^2+y^2+z^2+w^2 = a^2.$$

$\Rightarrow$ diff $x\,dx + \cdots = 0.$

dw

$\downarrow$

$$ds^2 = dx^2 + dy^2 + dz^2 + \cdots$$

$\Downarrow$

$\rho + \theta + \phi$

$$\underline{ds^2 = \frac{a^2}{a^2-r^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2}$$

line element.

$a \gg r \quad a \to \infty.$

$$\frac{a^2}{a^2-r^2} \approx \frac{\cancel{a^2}}{\cancel{a^2}} = 1.$$

$\Rightarrow$ 3D Euclidean spherical coordinates.

4 length & volumes.

length. $ds^2 = g_{ab}\, dx^a dx^b.$

$$L_{AB} = \int_{u_A}^{u_B} \left| g_{ab} \frac{dx^a}{du} \frac{dx^b}{du} \right|^{\frac{1}{2}} du.$$

w/ :

$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + \dots + g_{NN}(dx^N)^2$$

$$dV = \sqrt{|g_{11} g_{22}|}\; dx^1 dx^2 \cdot \quad 2D$$

> ND $\quad dV = \sqrt{|g_{11} g_{22} \cdots g_{NN}|}\; dx^1 dx^2 \cdots dx^N$

$$\Rightarrow dV = \sqrt{|g|}\; dx^1 dx^2 \cdots dx^N.$$

$$dx'^1 dx'^2 \cdots dx'^N = J\, dx^1 \cdots dx^N$$

$$J = \det\left(\frac{\partial x'^a}{\partial x^a}\right)$$

$$g'_{ab} = \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} g_{cd}$$

$$g' = g / J^2$$

($dV \Rightarrow$ invariance)

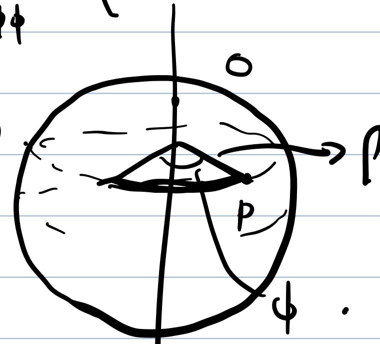
e.g. 2-sphere in $\mathbb{R}^3$.

$$ds^2 = \frac{a^2 d\rho^2}{\cdots} + \rho^2 d\phi^2$$

$a^2-\rho^2$

$$g_{\rho\rho} = \frac{a^2}{a^2-\rho^2} \qquad g_{\phi\phi} = \rho^2$$

The distance from center O to perimeter along curve $\phi = \text{const}$

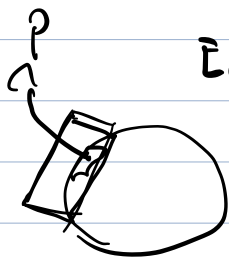


$$D = \int_0^R \frac{a}{(a^2-\rho^2)^{\frac{1}{2}}}\, d\rho = a \sin^{-1}\left(\frac{R}{a}\right)$$

$$A = \int_0^{2\pi}\int_0^R \frac{a^2}{\sqrt{a^2-\rho^2}}\, \rho\, d\rho\, d\phi = 2\pi a^2 \left(1-\sqrt{1-\left(\frac{R}{a}\right)^2}\right)$$

5. Local cartesian coordinate.

On Riemannian manifold ($ds^2>0$)
it is generally not possible to choose coordinate s.t. the line element takes Euclidean form at every point (ds^2)



We can always find coordinate s.t. at P

$$g_{ab}(P) = \delta_{ab} \quad \text{and} \quad \left.\frac{\partial g_{ab}}{\partial x^c}\right|_P = 0.$$

$$g_{ab} = \delta_{ab} + O[(x-x_p)^2]$$

$\Rightarrow$ local approx.

6. Riem ($ds^2 > 0$) $\Rightarrow$ Pseudo-Riem ($ds^2 < 0$)

$$g_{ab}(p) = \eta_{ab}.$$

$$\eta_{ab} = \mathrm{diag}(\pm 1, \pm 1, \pm 1, \dots)$$

$$ds^2 = \overset{+1}{d(ct)^2} \overset{-1}{- dx^2} \overset{-1}{- dy^2} \overset{-1}{- dz^2}$$

in Minkowski spacetime.